UNIVERSITÀ DEGLI STUDI DI MILANO

Relative $\mathbb{A}^1$-Contractibility of Smooth Schemes

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Open Poincaré Conjecture

Is $\mathbb{R}^n$ the unique open contractible smooth n-manifold?

Theorem: (Siebenmann, Stallings, et al. 1960's)

A. For $n < 3$, $\mathbb{R}^n$ is unique such manifold,
- From $n \geq 3$, there are prototypes of Whitehead manifolds $\mathcal{W}$ which are contractible but *not* homeomorphic to $\mathbb{R}^n$,

B. For $n \geq 3$, $\mathbb{R}^n$ is unique up to simply connectedness at infinity π_1^∞,

$$\pi_1^\infty(\mathbb{R}^n) \simeq \mathbb{S}^{n-1} \quad \pi_1(\mathbb{S}^{n-1}) = 0, \quad n \geq 3$$

- $\pi_1^\infty(\mathcal{W}) \neq 0$ as $\mathcal{W} \backslash C$ necessarily contains a (solid) torus for any compact $C \subset \mathcal{W}$.

Can we characterize $\mathbb{A}_k^n$ among smooth "contractible" schemes?

What is Motivic Homotopy Theory?

A homotopy theory for smooth separated S-schemes of finite type Sm_S with $I = \mathbb{A}^1$.

Established by Fabien Morel and Vladimir Voevodsky [MV99].

A *motivic S-space* $\mathcal{X}$ is a simplicial presheaf that satisfies

a. Descent via L_{Nis}: inverting all Nisnevich coverings $\check{C}(U_\bullet) \to X$,

b. $\mathbb{A}^1$-localization via $L_{\mathbb{A}^1}$: inverting all maps $\{\mathrm{pr}_1 : X \times \mathbb{A}^1 \to X : X \in Sm_S\}$

The category of (unstable) motivic S-spaces is then

$$Spc_S := L_{\mathbb{A}^1}(L_{Nis}(Pshv(Sm_S)))$$

Any scheme $X \in Sm_S$ defines a motivic space via the Yoneda embedding

$$\mathcal{X}(-) := Hom_{Sm_S}(-, X) : Sm_S^{op} \to sSet$$

$\mathbb{A}^1$-contractibles, Fiber spaces and ZLT bundles

- An S-scheme $f : X \to S$ is *$\mathbb{A}^1$-contractible* if f is an $\mathbb{A}^1$-weak equivalence in Spc_S.
- An S-scheme $p : X \to S$ is an *$\mathbb{A}^n$-fiber space* if f is smooth of finite presentation with all its fibers
$$p^{-1}(s) \cong \mathbb{A}^n_{\kappa(s)}$$
for all $s \in S$.
- p defines a *Zariski locally trivial $\mathbb{A}^n$-bundle* if $\forall s \in S$, $\exists$ an Zariski open $U \subset S$ such that
$$p^{-1}(U) := U \times_S X \xrightarrow{\cong} U \times \mathbb{A}^n$$
as U-schemes.

Examples: Affine n-spaces, Cusp $\{x^p = y^q\}$ with $(p,q) = 1$, Vector bundles and ZLT bundles with $\mathbb{A}^1$-contractible fibers.

Zariski Cancellation Problem (ZCP)

Over a field k, does $X \times \mathbb{A}^1_k \cong \mathbb{A}^{n+1}_k$ imply $X \cong \mathbb{A}^n_k$?

- True up to $n \leq 2$ for all k
- False for $n \geq 3$ over char $k > 0$
- Open for $n \geq 3$ over char $k = 0$

Characterization of Affine spaces $\mathbb{A}_k^n$

Is $\mathbb{A}_k^n$ the unique $\mathbb{A}^1$-contractible smooth affine scheme over a field k?

A variety V is *exotic* if $V \cong k^d$ topologically, but $V \ncong \mathbb{A}_k^d$ algebraically.

- True for $n = 0, 1$ over all k; Affine'ness not required: [DMØ],
- True for $n = 2$ over char $k = 0$ [Choudhury-Roy]; Affine'ness not required and extension to perfect fields: [DMØ], Is there a non-trivial k-form of $\mathbb{A}^2$ that is $\mathbb{A}^1$-contractible?
- False for $n \geq 3$ - Generalized Koras-Russell varieties [Dubouloz-Ghosh] and Asanuma varieties,
- False for $n \geq 4$ - arbitrary family of non-isomorphic exotic quasi-affines [Asok-Doran].

For $d = 3$, does $\mathbb{A}^1$-contractibility imply affine?

Hunt down: Exotic varieties

Koras-Russell threefolds over char $k = 0$

$$\mathcal{K} := \{X^m Z = X + Y^r + T^s\} \subseteq \mathbb{A}_k^4$$

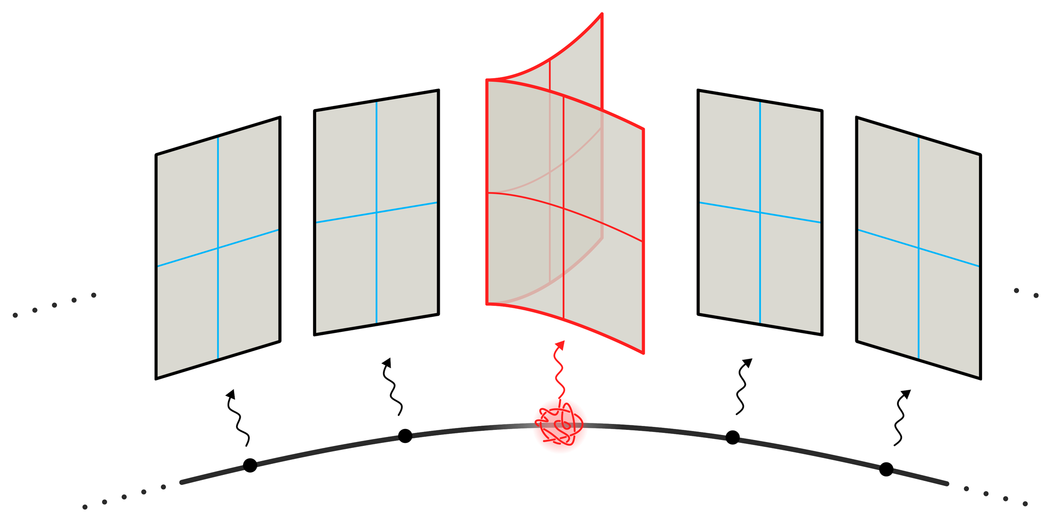
where $m, r, s \geq 2$ integers with r, s coprime.

- ⋆ $\mathcal{K} \cong_{top} \mathbb{R}^6$ [Dimca, Ramanujam],
- ⋆ $\mathcal{K} \ncong_{alg} \mathbb{A}^3_{\mathbb{C}}$ [Makar Limanov],

$$ML(\bigstar) := \bigcap_{\partial \in LND(\bigstar)} Ker\ \partial$$

$$ML(\mathbb{A}^3_{\mathbb{C}}) = \mathbb{C}, \quad ML(\mathcal{K}) = \mathbb{C}[X]$$

- ⋆ $\mathcal{K}$ is $\mathbb{A}^1$-contractible ([HKØ16], [DF18]). Using étale locally trivial $\mathbb{A}^1$-bundles and $\mathbb{A}^1$-Brouwer degree valued in Milnor-Witt K-theory K_*^{MW},
- ⋆ The map $\mathrm{pr}_x : \mathcal{K} \to \mathbb{A}^1_x$ has all closed fibres $\cong \mathbb{A}^2$ but the generic fiber $\cong \mathbb{A}^1 \times Cusp$; So, pr_x is not an $\mathbb{A}^2$-fiber space whence cannot be a ZLT $\mathbb{A}^2$-bundle.



Is $\mathcal{K}$ cancellative? i.e., is $\mathcal{K} \times \mathbb{A}^1_k \cong \mathbb{A}^4_k$?

Asanuma-Gupta varieties over char $F = p > 0$

$$\mathcal{A} := \{X^m\, Z = f(Y, T)\} \subseteq \mathbb{A}^4_F$$

where $m, e, s \geq 2$ integers and $f(Y,T) := T + Y^{p^e} + T^{sp}$ such that $sp \nmid p^e$ and $p^e \nmid sp$ is the non-trivial line.

- $\mathcal{A} \ncong \mathbb{A}^3_F$ but $\mathcal{A} \times \mathbb{A}^1_F \cong \mathbb{A}^4_F$ (non cancellative!)
- $\mathrm{pr}_x : \mathcal{A} \to \mathbb{A}^1_F$ is an $\mathbb{A}^2$-fiber space but not a Zariski locally trivial $\mathbb{A}^2$-bundle [Asa87]
- pr_x is an $\mathbb{A}^1$-weak equivalence in $Spc_{\mathbb{A}^1_F}$ ($\mathbb{A}^1$-contractible!).

Relative $\mathbb{A}^1$-contractibility over arbitrary base schemes, [DMØ]

Are $\mathbb{A}^1$-contractible smooth schemes necessarily ZLT $\mathbb{A}^n$-bundles?

Let $f : \mathcal{X} \to S$ be a smooth scheme of finite type of relative dimension d over a Noetherian scheme S of finite Krull dimension. Then:

Relative dimension $d = 0$ (S arbitrary)

★ $\mathcal{X}$ is $\mathbb{A}^1$-contractible if and only if f is an isomorphism,
Separated étale S-schemes are $\mathbb{A}^1$-rigid!

Relative dimension $d = 1$ (S normal)

★ $\mathcal{X}$ is $\mathbb{A}^1$-contractible if and only if f is a ZLT $\mathbb{A}^1$-bundle,
$\mathcal{X} \cong \mathbb{A}^1_S \iff \Omega_f \cong \mathcal{O}_{\mathcal{X}}$

Relative dimension $d = 2$ (S Dedekind with char $\kappa_S = 0$)

★ For f affine: $\mathcal{X}$ is $\mathbb{A}^1$-contractible if and only if f is a ZLT $\mathbb{A}^2$-bundle
For S affine, $\mathcal{X} \cong \mathbb{A}^2_S \iff \omega_f \cong \mathcal{O}_{\mathcal{X}}$
False if char $\kappa_S > 0$: e.g., Asanuma-Gupta varieties!

★ Motivic homotopy does not detect ZLT $\mathbb{A}^d$-bundles for $d \geq 3$!

$\mathbb{A}^1$-contractibility is a pointwise phenomenon - both stably and unstably!

For $X \in Spc_S$ with the inclusion $i : \{s\} \hookrightarrow S$, X is $\mathbb{A}^1$-contractible in $Spc_S \iff i_s^*(X)$ is $\mathbb{A}^1$-contractible in $Spc_{\kappa(s)}$.

Some major goals of the upcoming article [Mad].

Koras-Russell over Arithmetic Schemes

The smooth affine variety $\mathcal{K} \to Spec\ \mathbb{Z}$ is $\mathbb{A}^1$-contractible in $\mathcal{H}(\mathbb{Z})$.

- Extend to (Dedekind) schemes with perfect residue fields,
- Provides a distinct family of exotic threefolds in mixed characteristics,
- Generalized deformed Koras-Russell bundles over (certain) Dedekind schemes,
- Potential obstruction to the ZCP in the relative setting.

Generalized Motivic Spheres

Over a (reasonable) scheme S, when does

$$X \simeq_{\mathbb{A}^1} \mathbb{A}^n_S \backslash \{0\} \implies X \cong_S \mathbb{A}^n_S \backslash \{0\}?$$

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