Motivic invariants for real surfaces

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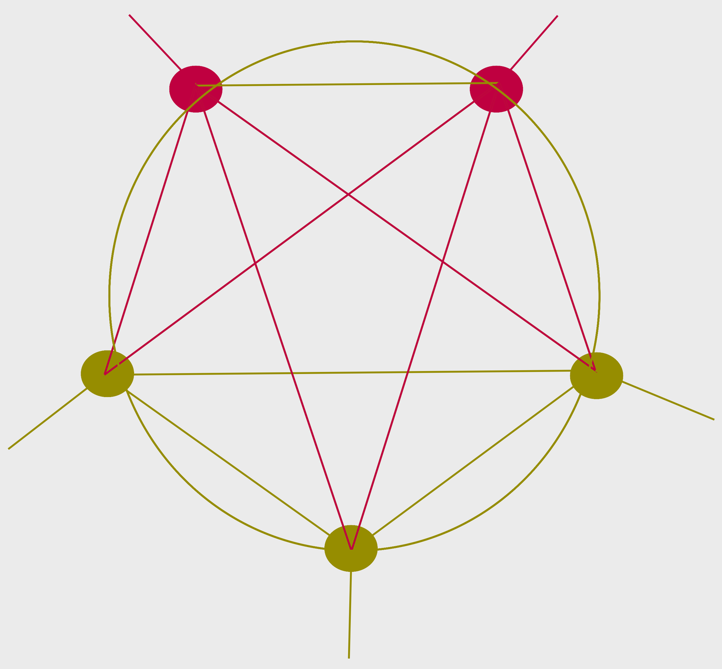
Motivic setup

To compute enumerative invariants of algebraic varieties, one often use the intersection theory. But most of the time we are restricted to working over an algebraically closed field. To put the known real invariants in the machinery, one can change the Chow ring of a variety X (as a cohomology ring $H^n(X, K_n^M)$) by a Chow-Witt ring $(H^n(X, K_n^{MW}(\mathcal{L})) = \widetilde{\mathrm{CH}}^n(X, \mathcal{L}))$. All the formal definitions live more generally in the stable $(\infty-)$category of motivic spectra with six functor formalism. $\mathcal{SH}(X)$ contains the spectra of representable $\mathbb{A}^1-$invariant (co)homology theories over X.

A drawing in degree 4

A degree 4 del Pezzo surface can be seen as the blow-up of $\mathbb{P}^2$ in 5 points in general position. The exceptional lines are given by the drawing :



If the blow-up is over $\mathbb{R}$ and two points are actually complex conjugated, we don't always have 16 lines but can have only 8 (or even less)! Actually, there was no known real count but one can find $8\langle 1\rangle + 8\langle -1\rangle \in GW(\mathbb{R})$ for the degree 4 del Pezzo surfaces.

The 27 complex / 3 real lines on the cubic

$$\begin{tikzcd}
 & \sigma_{f_3} \in \mathrm{Sym}^3(Q) \arrow[d] \\
V(f_3) \subseteq \mathbb{P}^3 \arrow[r, rightsquigarrow] \arrow[d] & \mathbb{G}(3,1) \arrow[d] \\
\{\text{lines of } V(f_3)\} \subseteq \mathbb{G}(3,1) \arrow[r, rightsquigarrow] & Z(\sigma_{f_3}) \subseteq \mathbb{G}(3,1)
\end{tikzcd}$$

Over $\mathbb{C}$, the class of $Z(\sigma_{f_3})$ in $\mathrm{CH}^4(\mathbb{G}(3,1))$ is independent of the choice of the global section and $[Z(\sigma_{f_3})] = c_4(\mathrm{Sym}^3(Q))$. Its degree is 27.
Over $\mathbb{R}$, this is false but using the more general Euler class and some motivic machinery, we recover :

$$\deg e(\mathrm{Sym}^3(Q)) = 15\langle 1\rangle + 12\langle -1\rangle \in GW(\mathbb{R}).$$

Milnor-Witt K-theory

The main difference comes from the addition of a new element η in degree -1 in the original Milnor K-theory. This change implies :

1. we now have quadratic data : $K_0^{MW}(k) \simeq GW(k)$ the group of quadratic forms over k,
2. we have to consider twists by invertible sheaves (i.e. orientations) to have the degree function $\widetilde{\mathrm{CH}}^{\dim X}(X, \omega_X) \to GW(k)$.

These changes and the fact that the degree takes into account traces from $\kappa_x \to k$ give enough data to have enumerative invariants over non algebraically-closed fields.

A motivic Euler class

The main class used is the Euler class defined for a rank r vector bundle $\mathcal{E} \to X$ as the composition :

$$\begin{tikzcd}
\mathbb{S}_X \arrow[r, "\eta"] \arrow[rrr, bend right, "e(\mathcal{E})"] & s^! s_! \mathbb{S}_X \arrow[r, "\mathfrak{p}_s"] & \Sigma^{\mathcal{E}} s^* s_* \mathbb{S}_X \arrow[r, "\epsilon"] & \mathrm{Th}_X(\mathcal{E})
\end{tikzcd}$$

$$c_r(\mathcal{E}) \in \mathrm{CH}^r(X) \qquad e(\mathcal{E}) \in \widetilde{\mathrm{CH}}^r(X, \det \mathcal{E})$$

Oriented intersection

The computation of intersection products in the Chow-Witt groups, even on the Grassmannian, is quite complicated. There are some ways to compute the Euler degree directly, or one can use the fiber product

$$\begin{tikzcd}
\widetilde{\mathrm{CH}}^{\dim X}(X, \mathcal{L}) \arrow[r] \arrow[d] & \mathrm{CH}^{\dim X}(X) \arrow[d] \\
\mathrm{W}^{\dim X}(X, \mathcal{L}) \arrow[r] & \mathrm{Ch}^{\dim X}(X)
\end{tikzcd}$$

to recover the Chow-Witt data from classical intersection in Chow groups and computation over the Witt groups, which is easier to do.

Reasons to work on Grassmannians

Expressing the problem over a Grassmannian is interesting in the classical setup because with the splitting principle and Schubert calculus, one can easily compute the Chern classes of vector bundles. However, the splitting principle is not true in (Chow-)Witt rings. To go around this problem we can use some equivariant Witt cohomology and then do the Witt-valued Schubert calculus.

What about orientations

The computations over the Grassmannians are mostly of Euler classes of vector bundles. In Chow(-Witt) groups, we have to consider the orientability. That is the class of $\det \mathcal{E}$ in $\mathrm{Pic}(G)/2$ and it has to be the same as $\omega_G \simeq \mathcal{O}_G(-n)$. There is a problem for the degree 4 example : $\det(\mathrm{Sym}^2(Q) \oplus \mathrm{Sym}^2(Q)) \simeq \mathcal{O}(6)$, but $\omega_G \simeq \mathcal{O}_G(-5)$. To recover an orientability, we can modify the vector bundle with tensor products by $\mathcal{O}(l)$. However, the geometrical meaning of the what we are counting is then harder to understand.

Equivariant cohomology

To recover an expression of $\mathrm{Sym}^3(Q)$ depending on $e(Q)$ without the splitting principle can be achieved using the $N-$equivariant Witt cohomology, with N the normalization of the torus $\begin{pmatrix} t & 0 \\ 0 & t^{-1} \end{pmatrix}$ in SL_2. The equivariant Euler class lives then in the equivariant cohomology. But $rk(\mathrm{Sym}^3(Q)) = \dim(G)$, and the degree of $e_N(\mathrm{Sym}^3(Q))$ lives in $\mathrm{W}_0(BN)$ and this is a $W(k)-$module! The trace of $e_N(\mathrm{Sym}^3(Q))$ in $\mathrm{W}(k)$ corresponds to the usual degree. Using the presentation of the $\mathrm{W}(\mathrm{Gr}(4,2))$ as a $\mathrm{W}(k)-$algebra generated by $e(Q)$ and $e(\mathcal{S})$, one can find :

$$e(\mathrm{Sym}^3(Q)) = 3e(Q)^2 \in \mathrm{W}^4(\mathrm{Gr}(4,2)).$$

The $GW-$degree

With the classical intersection result $\deg(c_4(\mathrm{Sym}^3(Q))) = 27$ and the Witt result $\deg(e(\mathrm{Sym}^3(Q))) = 3\langle 1\rangle \in \mathrm{W}(\mathbb{R})$, we recover the claimed result by adding $12(\langle 1\rangle + \langle -1\rangle)$ to have a degree 27 quadratic form in $GW(\mathbb{R})$.