TOPICS IN POLYNOMIAL DERIVATIONS AND THEIR AUTOMORPHISM GROUPS: CLASS NOTES

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Abstract. These notes correspond to the Master Class session given at the Groups en Actions meeting, at Poitiers in May 2026. We cover some basic aspects of the classification of $\Bbbk$-derivations of $\Bbbk[x_1,\dots,x_n]$ up to conjugation by the automorphisms group $\mathrm{ACr}(n)$. In particular, we study the algebraic and geometric properties of the isotropy group $\mathrm{Aut}(D)$ of a simple derivation D. We keep the prerequisites on algebraic geometry at a minimum, but also give some insight on the schematic aspects of the problems.

1. Introduction

Let $\Bbbk$ be an algebraically closed field. It is well known that, if $n \geq 2$, one cannot endow the so-called *Affine Cremona Group* $\mathrm{ACr}(n)$, consisting of the automorphisms of the affine space $\mathbb{A}^n_{\Bbbk}$, with a canonical structure of variety, but $\mathrm{ACr}(n)$ can be seen as an ind-variety. Under this point of view, a lot of work has been done: for example, one may characterize when a closed subgroup $G \subset \mathrm{ACr}(n)$ inherits a structure of algebraic group — in this case we have a faithful action of G on the affine space.

Analogously, since $\mathrm{ACr}(n)$ does not support a canonical structure of group scheme, but can be seen as an affine ind-scheme, the same problems hold in this setting — and also a lot of work has been done (see Section 6 below).

On the other hand, let $\mathrm{Der}\big(\Bbbk[x_1,\dots,x_n]\big)$ be the vector space of the $\Bbbk$-derivations $D : \Bbbk[x_1,\dots,x_n] \to \Bbbk[x_1,\dots,x_n]$ — a $\Bbbk$-derivation is a linear map D satisfying the Leibniz rule $D(fg) = fD(g) + gD(f)$. As we will see, the vector space $\mathrm{Der}\big(\Bbbk[x_1,\dots,x_n]\big)$ is related to two geometric notions: those of *vector field* and *foliation* of the affine space $\mathbb{A}^n_{\Bbbk}$.

By thinking of the elements in $\mathrm{ACr}(n)$ as $\Bbbk$-algebra automorphisms of $\Bbbk[x_1,\dots,x_n]$, we see that $\mathrm{ACr}(n)$ acts by conjugation on $\mathrm{Der}\big(\Bbbk[x_1,\dots,x_n]\big)$ — this action has a geometric counterpart when we think of $\mathrm{ACr}(n)$ acting on $\mathbb{A}^n_{\Bbbk}$. If D is a derivation, then its isotropy group $\mathrm{Aut}(D) \subset \mathrm{ACr}(n)$, which necessarily codifies important properties of the conjugation class of D, inherits a structure of closed ind-subvariety of $\mathrm{ACr}(n)$, and therefore acts on $\mathbb{A}^n_{\Bbbk}$. It is natural to ask in which measure (the equivalence class of) D is determined by $\mathrm{Aut}(D)$, at least in the case where $\mathrm{Aut}(D)$ is an algebraic group (or more generally a group scheme).

This mini-course focuses in introducing some tools to deal with this type of problem. More precisely: first we explain how to endow $\mathrm{Aut}(D)$ with a structure of closed ind-subgroup of $\mathrm{ACr}(n)$ and propose some problems relating to that. Subsequently, we describe the situation in dimension 2, where all is quite known. Finally, we give some results in higher dimension, pertaining to the following two general questions:

- *What can we say about a derivation D for which its isotropy is an algebraic group?*
- *Conversely, if D is a derivation with additional properties (*e.g.* being locally nilpotent or simple, as in Definition 2.8), what can be said about its isotropy group?*

2. Basic concepts and results

Let B be a $\Bbbk$-domain (that is, a $\Bbbk$-algebra that is also a domain) over an algebraically closed field $\Bbbk$ of characteristic 0. We will moreover assume that B is of finite type (that is generated as an $\Bbbk$-algebra by a finite number of elements). We will adopt the following conventions.

Notations.

- B^* is the group of unities of B.
- $\mathrm{Frac}(B)$ is the field of fractions of B.
- $\mathrm{Aut}_\Bbbk(B)$ is the group of $\Bbbk$-linear automorphisms of B.
- $\mathrm{tr.deg}_A B$ is the transcendence degree of B over a $\Bbbk$-subalgebra $A \subset B$.

2.1. Derivations and their isotropy.

Definition 2.1. A $\Bbbk$-*derivation* of B is a linear map $D : B \to B$ such that satisfies the "Leibniz rule": $D(ab) = D(a)b + aD(b)$. We denote by $\mathrm{Der}_\Bbbk(B)$ the set of all $\Bbbk$-derivations of B. For simplicity we will refer to an element $D \in \mathrm{Der}_\Bbbk(B)$ only as "a derivation".

The *kernel* of a derivation D is the kernel of D as a linear transformation, we denote it as $\ker D$.

Lemma 2.2. *We have the following assertions:*

(a) *If $D \in \mathrm{Der}_\Bbbk(B)$, then $\ker D$ is an algebraically closed $\Bbbk$-subalgebra of B such that* $\mathrm{Frac}(A) \cap B = A$.

(b) $\mathrm{Aut}_\Bbbk(B)$ *acts on* $\mathrm{Der}(B)$ *by conjugation.*

(c) *Given $b \in B$ and $D, E \in \mathrm{Der}_\Bbbk(B)$, then $D + E$, bD and $[D, E] := DE - ED$ belong to* $\mathrm{Der}_\Bbbk(B)$. *In particular,* $\mathrm{Der}_\Bbbk(B)$ *is a B-module and a Lie algebra with bracket* $[\ ,\ ]$.

Proof. Assertions (b) and (c) are straightforward. To prove (a) take $b \in B$ algebraic over $\ker D$ and consider a dependence relation $a_n b^n + a_{n-1} b^{n-1} + \cdots + a_1 b + a_0 = 0$, $a_0, \ldots, a_n \in$

$\ker D$, with $a_n \neq 0$ and n minimal. Then $(na_n b^{n-1} + (n-1)a_{n-1}b^{n-2} + \cdots + a_1)D(b) = 0$, and we deduce $D(b) = 0$, so $\ker D$ is algebraically closed in B.

On the other hand, if $c, d \in A$, with $d \neq 0$ and $c/d = b \in B$, then $D(c) = 0 = dD(b)$, which completes the proof. $\square$

Remark 2.3. One may see that the property stated as assertion (a) above characterizes $\Bbbk$-subalgebras of B defined as the kernel of a derivation (see for example [17, Thm. 4.1.4]).

Example 2.4. If $B = \Bbbk[x_1, \ldots, x_n]$, then $\mathrm{Der}_{\Bbbk}(B) = \bigoplus_{i=1}^n B\frac{\partial}{\partial x_i}$, where $\frac{\partial}{\partial x_i} = \partial_{x_i}$ denotes the formal derivative with respect to x_i, $i = 1, \ldots, n$.

Exercise 2.5.
(1) Prove that for any $f \in B$, then $D(fB) \subset fB \Leftrightarrow D(f) = \lambda f$, for some $\lambda \in B$ (see Definition 2.31 below).
(2) If B is a UFD and $f = f_1 \cdots f_r$ with the f_i's irreducible, then $D(f) \in fB$ implies $D(f_i) \in f_i B$ for all i.

The above exercise gives a necessary and sufficient condition for a principal ideal to be *stable*:

Definition 2.6. If $D \in \mathrm{Der}_{\Bbbk}(B)$, an ideal $I \subset B$ is said to be *D-stable* (or stable if there is no possible confusion) if $D(I) \subset I$.

Remark 2.7. If B is of finite type, then B is the $\Bbbk$-algebra $\Bbbk[X]$ of regular functions of an (irreducible since B is a domain) affine variety X. Moreover, there is a natural isomorphism from the automorphisms group $\mathrm{Aut}(X)$ of X to $\mathrm{Aut}_{\Bbbk}\big(\Bbbk[X]\big)$, given by the pull-back (that is, the pre-composition) by F: $F \mapsto F^* = -\circ F$. This isomorphism induces in turn an action of $\mathrm{Aut}(X)$ on $\mathrm{Der}_{\Bbbk}\big(\Bbbk[X]\big)$ defined as $F \cdot D = (F^{-1})^* \circ D \circ F^*$.

$$\begin{array}{ccc} \Bbbk[X] & \xrightarrow{F\cdot D} & \Bbbk[X] \\ {\scriptstyle F^*}\downarrow & & \downarrow{\scriptstyle F^*} \\ \Bbbk[X] & \xrightarrow[D]{} & \Bbbk[X] \end{array}$$

Thus, the orbit of a derivation D can be thought of as the different expressions of D "up to a change of variables" by an automorphism, giving place to the following:

Problems: (1) Classify derivations up to conjugation.

(2) Describe the orbit of a given derivation D — or equivalently, describe the isotropy subgroup $\mathrm{Aut}(D)$.

In all generality, these problems are very difficult; therefore in the literature they are usually restricted to special classes of derivations of recognized relevance as the ones introduced in the definition below (see next section).

Definition 2.8 (Some clases of derivations)**.** Let $D \in \mathrm{Der}_{\Bbbk}(B)$.

• D is said to be *simple* if for any ideal $I \subset B$, $D(I) \subset I$ implies $I = 0$ or $I = B$ — that is, the only D-stable ideals are the trivial ones.

• D is said to be *locally finite* if for any $f \in B$ the subset $\{f, D(f), D^2(f), \dots, D^n(f), \dots\}$ generates an finite dimensional vector subspace of B.

• D is said to be *locally nilpotent* if for any $f \in B$ there exists $n \geq 1$ such that $D^n(f) = 0$; in particular, in that case D is locally finite. We denote by $\mathrm{LND}(B) \subset \mathrm{Der}_{\Bbbk}(B)$ the $\Bbbk$-subspace of locally nilpotent derivations of B.

2.2. **Some algebraic and geometric related notions.**

In this section we describe some algebro-geometric properties of either B or the associated affine variety, related to properties of a given derivation $D \in \mathrm{Der}_{\Bbbk}(B)$. We begin by describing the correspondence between derivations and vector fields.

2.2.1. *Vector field and foliation associated with a derivation.*

Example 2.9. Assume for a moment that $\Bbbk = \mathbb{C}$ and let $z \in \mathbb{A}^n = \mathbb{A}^n_{\mathbb{C}}$ and $v \in \mathrm{T}_z\mathbb{A}^n = \mathbb{A}^n$. Then the directional derivative along v

$$D_z(f) = \frac{\partial f}{\partial v}(z) \text{ for any } f \in \mathbb{C}[x_1, \dots, x_n]$$

defines a *pointed derivation* at $z \in \mathbb{C}^n$, that is:

$$D_z(fg)(z) = f(z)\big(D_z(g)\big)(z) + \big(D_z(f)\big)(z)g(z) \text{ for all } f, g \in \mathbb{C}[x_1, \dots, x_n].$$

Now, if $\chi : \mathbb{C}^n \to \mathbb{C}^n$, $\chi(z) = \big(g_1(z), \dots, g_n(z)\big)$, $g_i \in \Bbbk[x_1, \dots, x_n]$ is an polynomial vector field, then $D : \Bbbk[x_1, \dots, x_n] \to \Bbbk[x_1, \dots, x_n]$, given by

$$D = \sum_i g_i \frac{\partial}{\partial_{x_i}},$$

is a derivation of $\Bbbk[x_1, \dots, x_n]$, such that $D(f)(z) = D_{\chi(z)}(f)$.

The reader with some background in algebraic geometry will recognize the above example as a particular case of the following construction:

Example 2.10. If X is an affine algebraic variety then $v \in T_xX$ (the tangent space of X at $x \in X$), identifies with derivation pointed at x, that we denote $D_v : \Bbbk[X] \to \Bbbk$. In this context, when X is smooth, a vector field of X (*i.e.* a global section of its tangent bundle) corresponds to a derivation $D : \Bbbk[X] \to \Bbbk[X]$.

Exercise 2.11. If $B = \mathbb{C}[x_1, \dots, x_n]$ and $f \in B$ is irreducible, prove that $D(f) \in fB$ if and only if its corresponding vector field χ_D is tangent to the hypersurface $f = 0$ at non-singular points of it (see Exercise 2.4).[1]

[1]Since we are working over the complex numbers, the reader is invited to choose either the "differential geometry" or the "algebraic geometry" approach, according to their background.

Remark 2.12. If $D \in \mathrm{Der}_{\Bbbk}\big(\Bbbk[x_1, \dots, x_n]\big)$ and $\chi_D = (g_1, \dots g_n)$ is the associated (polynomial) vector field, then a maximal ideal $\mathfrak{m} = \langle x_1 - \alpha_1, \dots, x_n - \alpha_n \rangle$ is D-stable if and only if $D(x_i - \alpha_i) = D(x_i) = g_i$ vanishes at $\alpha = (\alpha_1, \dots, \alpha_n)$, that is if and only if χ_D vanishes at α.

More generally, if B is of finite type and $D \in \mathrm{Der}_{\Bbbk}(B)$ stabilizes an ideal $I \subset B$, then D stabilizes all minimal prime $\mathfrak{p}$ associated to I ([21]).

In the particular case of $B = \mathbb{C}[x_1, \dots, x_n]$, we have that the inclusion $D(\mathfrak{p}) \subset \mathfrak{p}$ is equivalent to χ_D being tangent to the subvariety $\mathcal{V}(\mathfrak{p}) \subset \mathbb{C}^n$ at non-singular points of it. In this case we deduce that D is simple if and only if the *singular foliation* on $\mathbb{C}^n$ defined by χ_D has no [2] invariant algebraic subsets (in particular, it is non-singular).

2.2.2. *A regularity criterion.*

The existence of simple derivations detects regularity of a $\Bbbk$-algebra:

Proposition 2.13. *If $(B, \mathfrak{m})$ is the localization of a $\Bbbk$-algebra of finite type, then B is regular if and only if $\mathrm{Der}_{\Bbbk}(B)$ contains a simple derivation (*[21] *and* [9]*).*

2.2.3. *Simplicity of differential Ore extensions.*

Given a $\Bbbk$-algebra B we define a *skew polynomial ring* $B[t; D]$,[3] where the new operation is determined by $tb - bt = D(b)$, for all $b \in B$. One may prove that D is simple if and only if $B[t; D]$ has no nontrivial bilateral ideals (i.e. $B[t; D]$ is a simple ring; [8, Prop. 2.1]).

2.2.4. *Affine algebraic groups.*

Definition 2.14. An *algebraic group* is an algebraic variety G with an associative product (morphism of algebraic varieties) $m : G \times G \to G$ with neutral element 1_G, and an inverse morphism $\iota : G \to G$ that is a morphism of algebraic varieties.

If G is moreover an affine algebraic variety, we say that G is an *affine algebraic group*.

The definition of group $\Bbbk$-scheme is a little more involved, see Definition 6.8 below.

Examples 2.15. (1) $\mathrm{GL}_n(\mathbb{C})$ is an affine algebraic group. Its algebra of regular functions is $\Bbbk[a_{11}, \dots, a_{nn}]_{\det}$, because $\mathrm{GL}_n(\mathbb{C})$ is the open subset of the $n \times n$-matrices affine space, consisting of the matrices with non-zero determinant.

(2) $(\mathbb{C}, +)$, is an affine algebraic group, since $+ : \mathbb{C} \times \mathbb{C} \to \mathbb{C}$, $+(z, z') = z + z'$ and $- : \mathbb{C} \to \mathbb{C}$ are polynomials. We call this group the *additive group* and we denote it by $\mathbb{G}_a$.

[2] For any non-singular point $x \in X$, there is a neighborhood $U \ni x$ and a holomorphic submersion $h : U \to \Delta$ onto a disc $\Delta \subset \mathbb{C}$ whose fibers are tangent to χ_D.

[3] Such an extension is also known as an Ore extension after being introduced by Ore in [18].

(3) The n-dimensional *algebraic torus* $\mathbb{T}_n = \mathbb{C}^* \times \cdots \times \mathbb{C}^*$ is also an affine algebraic group.

The following well known result will allow us to work with affine algebraic groups while keeping the use of algebraic geometry results to a minimum.

Theorem 2.16 (Demazure and Gabriel, 1970)**.** *Let G be an affine algebraic group. Then there exists $n \geq 0$ and a closed immersion of algebraic groups $\varphi : G \to \mathrm{GL}_n(\Bbbk)$.*

Proof. The original statement and proof is about groups schemes, and can be found in [3]. See for example [5] for a proof for algebraic groups. □

Exercise 2.17. Describe $\mathbb{G}_a$ and $\mathbb{T}_n$ as closed subgroups of some $\mathrm{GL}_\ell(\Bbbk)$.

It is well know — and unfortunately outside the scope of this mini-course — that $T_{1_G}G$, the tangent space of an affine algebraic group at the neutral element, admits a structure of Lie algebra. This structure is related to the notions of *invariant vector fields* and *derivations*, see for example [10, Chapter VII] and [5, Chapter 5].

Definition 2.18. Let G be an algebraic group. A *regular action* of G on the algebraic variety X is an action $\varphi : G \times X \to X$ that is a morphism of algebraic varieties.

If $x \in X$, the *isotropy group* of x, noted by G_x is the — necessarily closed subgroup of G consisting of the elements fixing x:

$$G_x = \{g \in G : g \cdot x = x\}.$$

An action $G \times X$ is *free* is $G_x = \{1_G\}$ for all $x \in X$.

Examples 2.19.
1. If $v \in \mathbb{A}^n$, then the *action of $\mathbb{G}_a$ by translations* $\mathbb{G}_a \times \mathbb{A}^n \to \mathbb{A}^n$, $a \cdot x = t_{av}(x) = x + av$ is regular and free (if $v \neq 0$).
2. A *finite dimensional representation of G* is given by a morphism $G \to \mathrm{GL}(V)$, where V is a finite dimensional vector space. The corresponding action by linear transformations $G \times V \to V$ is regular, but not free, since $G_0 = G$.
3. In particular, the action of $\Bbbk^*$ on $\mathbb{A}^n$ by homotheties is regular. If $x \neq 0$, the $(\Bbbk^*)_x = \{1\}$.

Remark 2.20. Let $G \subset \mathrm{GL}_\ell(\Bbbk)$ be an affine algebraic group. Then an action of G on $\mathbb{A}^n$ is given by "polynomials" $g_1, \dots, g_n \in \Bbbk[a_{11}, \dots, a_{\ell\ell}]_{\det}[x_1, \dots, x_n]$: if $g \in G$ and $x = (x_1, \dots, x_n) \in \mathbb{A}^n$, then

$$g \cdot x = \big(g_1(g, x), \dots g_n(g, x)\big)$$

The polynomials g_i must satisfy additional conditions (imposed by the fact that we have an action).

Remark 2.21. It is well know that to give an affine algebraic group is equivalent to give a Hopf algebra (that as algebra is of finite type, without nilpotents), the association given by $G \mapsto \Bbbk[G]$. Under this point of view, to give an action on the affine algebraic variety X is equivalent to give a structure of $\Bbbk[G]$-comodule on $\Bbbk[X]$. See for example [5] for more details.

2.2.5. *Locally nilpotent derivations and $\mathbb{G}_a$-actions.*

The correspondence of derivations with tangent vectors fields applied in the context of $\mathbb{G}_a$-actions on affine algebraic varieties gives a pretty forward relationship of these actions with locally nilpotent derivations as follows.

Let $\psi : \mathbb{G}_a \times X \to X$ be a (non trivial) regular action of $\mathbb{G}_a$ over the irreducible affine variety X. Then ψ induces a right action on $\Bbbk[X]$, given by $(f \cdot t)(x) = f(t \cdot x)$, and

$$D(f) = \frac{d}{dt}\Big|_{t=0} f\big(t \cdot x\big),\ \forall f \in \Bbbk[X]$$

is a LND (see [6, Section 1.5]).

Conversely, if $D \in \mathrm{LND}\big(\Bbbk[X]\big)$, the formal series $e^{tD} = \sum_{n=0}^{\infty} \frac{t^n D^n}{n!}$ defines a morphism

$$F : \Bbbk[X] \to \Bbbk[t] \otimes \Bbbk[X] \cong \Bbbk[X][t],$$

that in turn induces a morphism $\psi : \mathbb{G}_a \times X \to X$, that is a regular action — again, $e^{tD} : \Bbbk[X] \to \Bbbk[X]$ is an automorphism, with inverse e^{-tD}, for all $t \in \Bbbk$.

Example 2.22. Let $v = (v_1, \ldots, v_n) \in \mathbb{A}^n$ and $G_a \subset \mathrm{ACr}(n)$ the sub-group of translations by tv:

$$\{x \mapsto t \cdot x = x + tv : t \in \Bbbk\} = \big\{(x_1 + tv_1, \ldots, x_n + tv_n) : t \in \Bbbk\big\} \subset \mathrm{ACr}(n).$$

Then

$$D(x_i) = \frac{d}{dt}\Big|_{t=0} (x_i + tv_i) = v_i\ ,\ \ i = 1, \ldots, n$$

and

$$D = \sum v_i \partial_{x_i}.$$

Conversely, if $D = \sum_i v_i \partial/\partial_{x_i}$, then D is locally nilpotent and

$$e^{tD}(x_i) = x_i + tv_i.$$

Therefore the corresponding $\mathbb{G}_a$-action is

$$t \cdot (x_1, \ldots, x_n) = (x_1 + tv_1, \ldots, x_n + tv_n)$$

as expected.

Example 2.23. Consider now the derivation $D = y\partial_x \in \mathrm{Der}\big(\Bbbk[x, y]\big)$. Then

$$e^{tD}(x) = x + yt,\ \ e^{tD}(y) = y.$$

Therefore e^{tD} induces the morphism $p(x, y) \mapsto p(x + ty, y)$, and we obtain the regular action

$$\varphi\big(t, (x, y)\big) = t \cdot \left(\begin{smallmatrix} x \\ y \end{smallmatrix}\right) = \left(\begin{smallmatrix} 1 & t \\ 0 & 1 \end{smallmatrix}\right) \left(\begin{smallmatrix} x \\ y \end{smallmatrix}\right) = \left(\begin{smallmatrix} x+ty \\ y \end{smallmatrix}\right)$$

Since the points $(a, 0)$ are fixed by this action, it follows that φ is not conjugated to a translation.

Example 2.24. More generally, if D is a locally nilpotent derivation of $B = \Bbbk[X]$ and $f \in \ker D$, then fD is locally nilpotent too and we get a homomorphism of groups $F : (\ker D, +) \to \mathrm{Aut}_\Bbbk(B)$ given by $f \to e^{fD}$. If we fix $f \in \ker D$, then F restricts to $\Bbbk f \cong \mathbb{G}_a$, defining a regular action $\varphi_f : \mathbb{G}_a \times X \to X$. Notice that since B is a domain, then $\ker D = \ker(fD)$. Moreover,

$$\ker D = B^{\mathbb{G}_a} = \left\{ b \in B : e^{tfD} b = b \forall t \in \mathbb{G}_a \right\} \quad , \quad \forall f \in \ker D.$$

Exercise 2.25. Consider a so-called *triangular derivation* D of $\Bbbk[x_1, \dots, x_n]$, *i.e.* a derivation of the form $D = \sum_{i=1}^n a_i \frac{\partial}{\partial x_i}$, where $a_i \in \Bbbk[x_{i+1}, \dots, x_n]$ if $i < n$ and $a_n \in \Bbbk$.

(1) Prove that D is locally nilpotent (hint: induction on n).
(2) Describe the automorphism e^D.

2.3. Properties and examples.

Examples 2.26. (a). $D : \Bbbk[x] \to \Bbbk[x]$ is simple if and only if it is a non zero locally nilpotent derivation, if and only if $D = c d/dx$ for some $c \in \Bbbk^*$.

(b) If $D \in \mathrm{Der}_\Bbbk(\Bbbk[x_1, \dots, x_n])$ is simple, then $\ker D = \Bbbk$.

Theorem 2.27. *(Shamsuddin, 1979) Let $\delta : B \to B$ be a simple derivation and extend it to a derivation $D : B[y] \to B[y]$ as $D(y) = ay + b$, for some $a, b \in B$. Then D is simple if and only if there is no $h \in B$ such that $\delta(h) = ah + b$.*

Proof. See [17]. □

Example 2.28. If $B = \Bbbk[x, y]$, then $D = \partial_x + (1 + xy)\partial_y$ is simple.

Proposition 2.29. *Let $D \in \mathrm{LND}(B)$ with $\mathrm{tr.deg}_\Bbbk B = n$, and set $A = \ker D$. Then $\mathrm{tr.deg}_\Bbbk \mathrm{Frac}(A) = n - 1$. Moreover, we have:*

(a) *A is factorially closed in B (i.e. $ab \in A$ implies $a, b \in A$).*

(b) *If $f \neq 0$ and $D(f) = \lambda f$ for some $\lambda \in B$, then $\lambda = 0$.*

Proof. [22, Prop. 1.3.32] □

Remark 2.30. Note that if D is simple and $f \in \ker D$, then $fB = 0$ or $fB = B$, i.e. either $f = 0$ or $f \in B^*$. Then if, for example, $B = \Bbbk[x_1, \dots, x_n]$ with $n \geq 2$, then $D \in \mathrm{Der}_\Bbbk(B)$ can not be both locally nilpotent and simple.

Definition 2.31. Let $D \in \mathrm{Der}_\Bbbk(B)$. An eigenvector of D is a nonzero element $b \in B$ such that $D(f) = \lambda b$ for some $\lambda \in B$; such a λ is called the eigenvalue of f. If $B = \Bbbk[x_1, \dots, x_n]$ we will also refer to f as a *Darboux polynomial* for D.

Remarks 2.32. (a) If D is simple, then their eigenvectors are contained in B^*.

(b) If D is locally nilpotent, then $\ker D \setminus \{0\}$ is the set of its eigenvectors (Proposition 2.29(b)).

Example 2.33. Let $D = x_1x_2\frac{\partial}{\partial_{x_1}}$. Then x_1 is an eigenvector with eigenvalue x_2 and $\ker D = \mathbb{k}[x_2, \ldots, x_n]$. Let $f \notin \ker D$ be another eigenvector of D; write $f = \sum_{i=0}^{\ell} a_i x_1^i$ for some $a_0, \ldots, a_\ell \in \mathbb{k}[x_2, \ldots, x_n]$, with $\ell > 0$ and $a_\ell \neq 0$. Then $\lambda \in \mathbb{k}[x_2, \ldots, x_n]$ and it follows that $f = a_\ell x_1^\ell$ and $\lambda = \ell x_2$.

Notice that D is not locally finite since $\{x_1, D(x_1), D^2(x_1), \ldots\}$ is infinite and linearly independent over $\mathbb{k}$). In particular, D is not a LND.

Now we introduce the following number associated to an element $D \in \mathrm{Der}_{\mathbb{k}}(\mathbb{k}[x_1, \ldots, x_n])$: $\mathbb{e}(\mathrm{D})$ is the number of irreducible eigenvectors up to multiplication by an element in $\mathbb{k}^*$.

Exercise 2.34. Assume that B is a UFD and consider the subset $\mathbb{E}(D) \subset \mathrm{Spec}(B)$ of the D-stable principal prime ideals of B with height 1.

(a) Prove that if $\varphi \in \mathrm{Aut}_{\mathbb{k}}(B)$ and $D_1 = \varphi D \varphi^{-1}$, then there is a bijective correspondence between $\mathbb{E}(D)$ and $\mathbb{E}(D_1)$. Deduce that $\#\mathbb{E}(D)$ is an invariant of D under conjugation by $\mathbb{k}$-automorphism of B.

(b) Prove that if $B = \mathbb{k}[x_1, \ldots, x_n]$, then $\mathbb{e}(\mathrm{D}) = \#\mathbb{E}(\mathrm{D})$.

Theorem 2.35 (Darboux, Jouanolou). *If $D \in \mathrm{Der}_{\mathbb{k}}(\mathbb{k}[x_1, \ldots, x_n])$ is such that $\mathbb{e}(\mathrm{D}) = \infty$, then there exist $f, g \in \mathbb{k}[x_1, \ldots, x_n]$ such that*

$$d(f/g) = \frac{D(f)g - fD(g)}{g^2} = 0.$$

Proof. [12, Prop. 3.6.8.]. □

Exercise 2.36. If $f, g \in \mathbb{k}[x_1, \ldots, x_n]$ are relatively prime, then $D(f/g) = 0$ implies that all the elements of the form $\alpha f + \beta g$, where $\alpha, \beta \in \mathbb{k}$, are eigenvectors of D with the same eigenvalue. Deduce that if $\mathbb{k} = \mathbb{C}$ and $\mathbb{e}(\mathrm{D}) = \infty$, then the vector field v_D associated to D is tangent (at non singular points) to the members of the pencil generated by the curves $f = 0$ and $g = 0$.

3. Polynomial automorphisms

In this section we deal with the geometry of the Affine Cremona group and its canonical action on $\mathbb{A}^n$.

3.1. The affine Cremona group.

We begin by recalling a basic fact from algebraic geometry:

An automorphism $F \in \mathrm{Aut}_{\mathbb{k}}(\mathbb{k}[x_1, \ldots, x_n])$ is determined by $f_1 = F(x_1), \ldots, f_n = F(x_n)$, which defines in turn a polynomial automorphism $\mathbb{A}^n_{\mathbb{k}} \to \mathbb{A}^n_{\mathbb{k}}$, given by $x \mapsto (f_1(x), \ldots, f_n(x))$.

Conversely, any polynomial automorphism $\mathbb{A}^n_{\Bbbk} \to \mathbb{A}^n_{\Bbbk}$ is of this form: if $\psi : \mathbb{A}^n_{\Bbbk} \to \mathbb{A}^n_{\Bbbk}$ is given by $\psi(x_1,\dots,x_n) = \big(p_1(x_1,\dots,x_n),\dots,p_n(x_1,\dots,x_n)\big)$, then $\psi^* : \Bbbk[x_1,\dots,x_n] \to \Bbbk[x_1,\dots,x_n]$, $\psi^*(g) = g\circ\psi$ is an automorphism.

We will identify $\mathrm{Aut}_{\Bbbk}(\Bbbk[x_1,\dots,x_n])$ with $\mathrm{Aut}(\mathbb{A}^n_{\Bbbk})$ and denote it by $\mathrm{ACr}(n)$ (for *Affine Cremona group*).

The *degree* of $F \in \mathrm{ACr}(n)$ is, by definition, the positive integer

$$\deg F = \max\big\{\deg f(x_1),\dots,\deg f(x_n)\big\}.$$

Remark 3.1. The reader with some background in algebraic geometry has identified for sure the above identification as a special case of the equivalence between the category of affine algebraic varieties and the category of affine $\Bbbk$-algebras.

In order to endow $\mathrm{ACr}(n)$ with a geometric structure, we need to introduce the notion of *(affine) ind-variety*. Roughly speaking, an ind-variety is a topological space that can be filtrated by algebraic varieties in a compatible way:

Definition 3.2. Let X be a set such that there exists a countable filtration by a family of algebraic varieties X_i, $i \in \mathbb{N}$ — that is, $X_i \subset X_{i+1}$ for all $i \in \mathbb{N}$ and $X = \bigcup X_i$ —, such that for every $i \in \mathbb{N}$, $X_i \subset X_{i+1}$ is a closed subvariety. We endow X with the following topology $\mathcal{T}$: $Z \subset X$ is closed if and only if $Z \cap X_i$ is closed for all $i \in \mathbb{N}$. We say that $(X,\mathcal{T})$ is an *ind-variety* — and call $\mathcal{T}$ the *ind-topology*.

If moreover all the X_i are affine varieties, the we say that the ind-variety X is affine.

A function $f : X \to Y$ between ind-varieties $X = \bigcup X_i$ and $Y = \bigcup Y_j$ is a *morphism of ind-varieties* if for every i there exists j such that $f(X_i) \subset Y_j$ and $f|_{X_i} : X_i \to Y_j$ is morphism of algebraic varieties. Notice in particular that a morphism of ind-varieties is continuous.

See [14, Chapter IV] for a short introduction to the subject.

Remark 3.3. Of course, one should establish a notion of *compatible filtrations* and prove that different compatible filtrations give the same structure of ind-variety and similar results, that again, do not fall in the scope of this mini-course.

Example 3.4 ($\mathrm{ACr}(n)$ is an ind-group)**.** Coming back to $\mathrm{ACr}(n)$, consider the filtration $\mathrm{ACr}(n) = \bigcup_r \mathcal{A}_r$, where $\mathcal{A}_r = \big\{f \in \mathrm{ACr}(n); \deg f \leq r\big\}$. Then, by a theorem of Kambayashi ([13]) we know that $\mathcal{A}_r$ is an affine algebraic variety such that $\mathcal{A}_r \subset \mathcal{A}_{r+1}$ is a closed immersion for all $r \geq 1$; thus, $\mathrm{ACr}(n)$ is an affine ind-variety.

Moreover, the map $\eta : \mathrm{ACr}(n)\times\mathrm{ACr}(n) \to \mathrm{ACr}(n)$ defined by $\eta(\varphi,\psi) = \varphi\psi^{-1}$ satisfies the following property: for any r, s, there exists t such that $\eta(\mathcal{A}_r\times\mathcal{A}_s) \subset \mathcal{A}_t$ in such a way that η induces a morphism of algebraic varieties $\mathcal{A}_r \times \mathcal{A}_s \to \mathcal{A}_t$; therefore the product and the inversion are morphisms of ind-varieties: one says that $\mathrm{ACr}(n)$ is an *(affine) ind-group*.

Examples 3.5. (a) $\mathbb{Z} \subset \Bbbk$ is an ind-group.

(*b*) $\mathbb{A}^\infty = \bigoplus_1{}^\infty \Bbbk$ is an ind-variety: consider the filtration given by the partial sums $\bigoplus_1^n \Bbbk$. The addition and the opposite are morphisms of ind-varieties, therefore $\mathbb{A}^\infty$ is an ind-group.

(*c*) Let $G \subset \mathrm{ACr}(n)$ be a closed subgroup. If there exists r such that $G \subset \mathcal{A}_r$, then G is an algebraic group. Conversely, if G is an algebraic subgroup (with respect to the induced operation), then there exists r such that $G \subset \mathcal{A}_r$. These results follow directly from results on ind-varieties and the characterization of algebraic subgroups of $\mathrm{ACr}(n)$ by Kambayashi (see [13]).

(*d*) If $\varphi \in \mathrm{ACr}(n)$, the orbit map $\mathrm{ACr}(n) \to \mathrm{ACr}(n)$, $\psi \mapsto \varphi\psi$ defines an isomorphism of ind-varieties: indeed, since $\deg(\varphi\psi) \leq \deg(\varphi)\deg(\psi)$ that map induces a morphism of algebraic varieties $\mathcal{A}_r \to \mathcal{A}_{r+\deg(\varphi)}$. We deduce that conjugating by φ is also an isomorphism.

By Example 3.5, it is easy to determine if a given closed subgroup $G \subset \mathrm{ACr}(n)$ is algebraic or not — provided that we have an explicit description of its elements —:

Examples 3.6. (1) $\mathrm{GL}_n(\Bbbk) \subset \mathrm{ACr}(n)$ and

$$\mathrm{Aff}_n(\Bbbk) = \left\{F + T_v : F \in \mathrm{GL}_n(\Bbbk)\,,\ T_v(x) = x + v\,, v \in \mathbb{A}^n\right\} \subset \mathrm{ACr}(n)$$

are algebraic subgroups.

(2) $f(x,y) = \big(a_1x + b_1, a_2y + b_2(x)\big)$, $a_i \neq 0, b_1 \in \Bbbk$ is an automorphism of $\Bbbk[x,y]$.

(3) More generally, $f(x) = \big(a_1x_1{+}b_1, a_2x_2{+}b_2(x_1), a_3x_3{+}b_3(x_1,x_2), \dots, a_nx_n{+}b_n(x_1,...,x_{n-1})\big)$, $a_i \neq 0$, $b_1 \in \Bbbk$ is a *de Jonquières automorphism*.

Exercise 3.7. Prove that the group of the de Jonquières automorphisms of $\Bbbk[x_1,\dots,x_n]$, that we denote as Jon_n (Examples 3.6) is a closed subgroup of $\mathrm{ACr}(n)$. By the Kambayashi's result cited above, it is not algebraic. However, the filtration of Jon_n given by the de Jonquières automorphisms of degree lower or equal to r ($\mathrm{Jon}_n \cap \mathcal{A}_r$ is a filtration by affine algebraic subgroups.

Exercise 3.8. Prove that if $D \in \mathrm{Der}_\Bbbk(\Bbbk[x_1,\dots,x_n])$, then $\mathrm{Aut}(D)$ is a closed subgroup of $\mathrm{ACr}(n)$. Deduce that $\mathrm{Aut}(D)$ inherits a structure of affine ind-group.

Example 3.9. Consider a derivation of $\Bbbk[x,y]$ of the form $D = b\partial_y$, where $b \in \Bbbk[x,y]$. Then $\varphi = (f,g) \in \mathrm{Aut}(D)$ if and only if

$$b\partial_y f = 0,\ b\partial_y g = b(f,g).$$

Then $f \in \Bbbk[x]$. Notice that the Jacobian determinant of φ is constant, hence $\partial_x f \partial_y g \in \Bbbk^*$. We deduce $f = \alpha x + \beta$ and $g = \gamma y + P(x)$, where $\alpha,\beta,\gamma \in \Bbbk$, and $P \in \Bbbk[x]$ is such that $\gamma b(x,y) = b(\alpha x + \beta, \gamma x + P(x))$.

If $b \in \Bbbk[x]$, then

$$\mathrm{Aut}(D) = \{(\alpha x + \beta, \gamma y + P(x)); P \in \Bbbk[x], \gamma b(x) = b(\alpha x + \beta)\}.$$

This group contains automorphisms of arbitrary degree, so it is not an algebraic group.

If b depends on y one may prove that the degree of elements in $\operatorname{Aut}(D)$ is bounded (see Exercise below), thus $\operatorname{Aut}(D)$ is an algebraic group in this case.

Exercise 3.10. Prove the last assertion in Example 3.9 (hint: if it is not the case, then there would be a sequence $(\alpha_m x+\beta_m, \gamma_m y+P_m(x)) \in \operatorname{Aut}(D)$ with $\lim_{m\to\infty} \deg P_m = \infty$).

Examples 3.11. We are going to fix $B = \Bbbk[x_1, \dots, x_n]$.

(a) If $n = 2$, by a result of Rentschler ([20]) one knows that D is locally nilpotent if and only if it is conjugate to a derivation of the form $b\partial_y$ for some $b \in \Bbbk[x]$. We deduce that $\operatorname{Aut}(D)$ is not algebraic (c.f. Example 3.9).

(b) More generally, if D is locally nilpotent with arbitrary $n \geq 2$, then $\operatorname{Aut}(D)$ is not algebraic. Indeed, $\operatorname{Aut}(D)$ contains the so-called *exponential automorphisms* of the form e^{aD}, for $a \in \ker D$, and $\operatorname{tr.deg}_{\Bbbk}\operatorname{Frac}(\ker D) = n-1$. One may deduce $\lim_{r\to\infty} \dim(\operatorname{Aut}(D) \cap \mathcal{A}_r) = \infty$.

Problem. *Classify D for which* $\operatorname{Aut}(D)$ *is an algebraic group.*

3.2. $\mathbb{A}^n$ as a $\operatorname{ACr}(n)$-variety.

The notion of regular action of an algebraic group on an algebraic variety can be extended in the obvious way to ind-groups:

Definition 3.12. Let G be an ind-group and X an ind-variety. An action $\varphi : G \times X \to X$ is *regular* if φ is a morphism of ind-varieties.

In this section we want to prove that the canonical action of $\operatorname{ACr}(n)$ on $\mathbb{A}^n$ is indeed regular. In order to do so, we need to describe the structure of $\mathcal{A}_r$ as an affine algebraic variety. For this, we will use a result of Furter and Kraft, that exhibits $\operatorname{ACr}(n)$ as a locally closed subset of the ind-variety $\Bbbk[x_1, \dots, x_n]^n$.

Remark 3.13. Since $\Bbbk[x_1, \dots, x_n]$ is a $\Bbbk$-vector space, with countable basis, $\Bbbk[x_1, \dots, x_n]$ admits a canonical structure of ind-variety. It follows that the monoid of endomorphisms of $\Bbbk[x_1, \dots, x_n]$ — that correspond to the monoid of endomorphisms of $\mathbb{A}^n$, is also an ind-variety, since

$$\operatorname{End}\big(\Bbbk[x_1, \dots, x_n]\big) \cong \Bbbk[x_1, \dots, x_n]^n.$$

Notice that if we filter $\Bbbk[x_1, \dots, x_n]$ by the total degree, we get a filtration by finite dimensional vector spaces (corresponding to the coefficients of the polynomials).

Theorem 3.14 (Furter-Kraft, [7])**.** *Let $\operatorname{Dom}\big(\Bbbk[x_1, \dots, x_n]\big)$ be the monoid of dominant endomorphisms of $\mathbb{A}^n$ — recall that $f : \mathbb{A}^n \to \mathbb{A}^n$ is dominant if f has dense image. Then $\operatorname{Dom}\big(\Bbbk[x_1, \dots, x_n]\big)$ is open in $\operatorname{End}\big(\Bbbk[x_1, \dots, x_n]\big)$ and $\operatorname{ACr}(n) \subset \operatorname{Dom}\big(\Bbbk[x_1, \dots, x_n]\big)$ is a closed subgroup.*

In particular, $\mathcal{A}_r \subset \Bbbk[x_1, \dots, x_n]_r$ (the space of polynomials of total degree lower or equal to r) is locally closed.

Proof. See [**?**, Theorem 5.2.1] — the proof only involves the use of Gröbner bases and some basic linear algebra. ☐

Proposition 3.15. *The canonical action of* $\mathrm{ACr}(n)$ *on* $\mathbb{A}^n$*, given by* $\varphi : \mathrm{ACr}(n) \times \mathbb{A}^n \to \mathbb{A}^n$*,* $\varphi\big(f, x\big) = \big(f_1(x), \dots, f_n(x)\big)$*, where* $f = (f_1, \dots, f_n)$*, is a regular action.*

Proof. We need to prove that $\varphi|_{\mathcal{A}_r \times \mathbb{A}^n} : \mathcal{A}_r \times \mathbb{A}^n \to \mathbb{A}^n$ is a morphism of algebraic varieties. But this is straight-forward by the very definition of the action. ☐

4. Polynomial derivations in two variables

We are going to fix $B = \Bbbk[x, y]$. The main result of this section is a geometric characterization of the property of $\mathrm{Aut}(D)$ being algebraic. Let as first introduce a last notion.

Definition 4.1. A polynomial $f \in \Bbbk[x, y]$ is said to be *rectifiable* if there is $\varphi \in \mathrm{ACr}(2)$ such that $\varphi(f) = x$. In that case we write $f \sim x$.

Thus, a polynomial is rectifiable if upon a (non necessarily linear) change of coordinates, the set of zeroes $\mathcal{V}(f) = \{f = 0\} \subset \mathbb{A}^2$ is a straight line.

If f is an irreducible polynomial, then the subgroup of $\mathrm{ACr}(2)$ consisting of those elements that leave invariant the curve $f = 0$ is an algebraic group if and only if f is not rectifiable ([2, Thm. 1]). The "only if" part is easy:

Exercise 4.2. Prove that if f is rectifiable, then $\{\varphi; \varphi(f)/f \in \Bbbk^*\}$ is not algebraic (hint: by arguing as in Example 3.9 prove that the elements in that subgroup are conjugate to automorphisms of the form $(\alpha x, \gamma y + P(x))$).

Deduce that if f is rectifiable, then its stabilizer for the canonical $\mathrm{ACr}(2)$-action is not algebraic.

Theorem 4.3 ([19])**.** *If* $D \in \mathrm{Der}_{\Bbbk}(\Bbbk[x, y])$ *then* D *is locally nilpotent if and only if* $\mathrm{Aut}(D)$ *is not an algebraic group.*

Proof. (Very sketchy) We only need to prove the converse. Let us assume that $\mathrm{Aut}(D)$ is not an algebraic group. We are going to prove that D is locally nilpotent. We consider different cases depending on the values of the invariant $\mathbb{e}(\mathrm{D})$ (recall the notation $\mathbb{E}(D)$ introduced in Exercise 2.34):

Case 1. Assume $0 < \mathbb{e}(\mathrm{D}) < \infty$. Then there is a group homomorphism $\kappa : \mathrm{Aut}(D) \to \mathrm{Per}(\mathbb{E}(D))$, where $\mathrm{Per}(\mathbb{E}(D))$ is the group of permutations of $\mathbb{E}(D)$.

Suppose for a moment that $\mathbb{E}(D)$ does not contain rectifiable elements. Then $K := \ker \kappa$ is an algebraic subgroup of finite index in $\mathrm{Aut}(D)$, so there are $\varphi_1, \dots, \varphi_s \in \mathrm{Aut}(D)$ such that $\mathrm{Aut}(D) = \cup_{i=1}^{s} \varphi_i K$. Since $\{\deg \varphi; \varphi \in K\}$ is bounded and $\deg(\varphi_i \varphi) \leq \deg(\varphi_i) \deg(\varphi)$ we deduce that the degree of elements in $\mathrm{Aut}(D)$ is also bounded, i.e., that subgroup is algebraic, contradicting our assumption.

Now we know that $\mathbb{E}(D)$ contains a rectifiable element. Then D is conjugate to a derivation D_1 such that $D_1 = x^\ell a(x,y)\partial_x + b(x,y)\partial_y$ for some $\ell \geq 1$, where either $a = 0$ or x does not divide a. It suffices to prove that D_1 is locally nilpotent (Example 3.5(c)).

First of all let us note that $a = 0$. In fact, if $a \neq 0$ and $x \not| a$, we will see that $\operatorname{Aut}(D_1)$ is an algebraic group. In fact, it suffices to show that K is algebraic.

Since $\varphi(x)/x \in \mathbb{k}^*$ for all elements in K, by arguing as above we deduce that such an element writes as $\varphi = (\alpha x, \gamma y + P(x))$. Then

$$\begin{cases} \alpha^\ell x^\ell a(\alpha x, \gamma y + P(x)) = \alpha x^\ell a(x,y), \text{and} \\ b(\alpha x, \gamma y + P(x)) = P'(x) x^\ell a(x,y) + \gamma b(x,y). \end{cases}$$

As for Exercise 3.10 one proves that $\deg P$ admits a bound depending on a and b.

On the other hand, if $a = 0$ and $\partial_y b \neq 0$, Exercise 3.10 also proves that $\operatorname{Aut}(D_1)$ is an algebraic group. Hence $a = 0$ implies $b \in \mathbb{k}[x]$ and the result follows in this case.

Case 2. $\mathbb{e}(\mathrm{D}) = \infty$. Theorem 2.35 implies that the vector field associated to D is (generically) tangent to the curves of a pencil. By using birational geometry one may prove that $\operatorname{Aut}(D)$ is realized (up to conjugation) either as a subgroup of the automorphism group of a Hirzebruch surface or as a subgroup consisting of elements of the form $(\alpha x + \beta, \gamma y + P(x))$. Moreover, in the second case one may suppose that $D = b(x,y)\partial_y$. Since the automorphisms group of a Hirzebruch surface is an algebraic group, we conclude that we are in the second situation, and the assertion follows as before (for more details see [1, Prop. 2.10 and Cor. 2.12]).

Case 3. $\mathbb{e}(\mathrm{D}) = 0$. One may suppose $\mathbb{k} = \mathbb{C}$ and use the foliation associated to D to prove that $\operatorname{Aut}(D)$ is finite in this case ([19, Prop. 2.3]). □

Example 4.4. $D = a(y)\partial_x + y\partial_y \in \operatorname{Der}_{\mathbb{k}}(\mathbb{k}[x,y,z])$ is not locally nilpotent since $D^n(y) = y$ for all n. But $\operatorname{Aut}(D)$ is not algebraic because it contains $(x + P(z), y, z)$ for an arbitrary $P \in \mathbb{k}[z]$.

Problem. *Classify non locally nilpotent derivations in* $\mathbb{k}[x_1, \ldots, x_n]$ *whose isotropy is not algebraic, when* $n \geq 3$.

5. What about simplicity?

Fix $B = \mathbb{k}[x_1, \ldots, x_n]$. If $n = 2$, what we have observed at the end of the proof of Theorem 4.3 implies that $\operatorname{Aut}(D)$ is finite. In fact, one may prove that $\operatorname{Aut}(D) = 1$ in this case (see [15]).

Conjecture. *If D is simple, then* $\operatorname{Aut}(D)$ *is an algebraic group.*

Example 5.1. Consider the Jordan's example ([11])

$$D_n = (1 - x_1 x_2)\frac{\partial}{\partial x_1} + x_1^3 \frac{\partial}{\partial x_2} + \sum_{i=3}^{n} x_{i-1} \frac{\partial}{\partial x_i},$$

where $n \geq 3$, which is known to be a simple derivation. By [23, Thm. 2.4] we know that $\mathrm{Aut}_{\Bbbk}(D_n)$ is the subgroup of translations in the last coordinate.

Example 5.2. Let $\delta : \Bbbk[u,v] \to \Bbbk[u,v]$ be a simple derivation, as for example $\delta = \partial_u + (1+uv)\partial_v$. If $a_1, \ldots, a_n \in \Bbbk[u,v]$ are linearly independent polynomials such that the linear subspace generated by them intersects $\delta\big(\Bbbk[u,v]\big)$ trivially, then $D = \delta + \sum_{i=1}^n a_i \partial_{x_i}$ is a simple derivation ([16, Prop. 4.11]). Notice that $\mathrm{Aut}(D)$ contains the translation

$$(u, v, x_1, \ldots, x_n) \mapsto (u, v, x_1 + c_1, \ldots, x_n + c_n),$$

for all $(c_1, \ldots, c_n) \in \Bbbk^n$.

As every affine ind-group, $\mathrm{Aut}(D)$ admits a connected subgroup $\mathrm{Aut}(D)^0$ which is normal and such that $\mathrm{Aut}(D)/\mathrm{Aut}(D)^0$ is discrete ([7, Proposition 2.2.1]). We have the following result (see [16]):

Theorem 5.3. *Let $D \in \mathrm{Der}(\Bbbk[x_1, \ldots, x_n])$ be a simple derivation. Then the following assertions hold:*

(*a*) $\mathrm{Aut}(D)^0$ *is a unipotent algebraic group of dimension* $\leq n-2$.

(*b*) *If $G \subset \mathrm{Aut}(D)$ is an (non necessarily connected) algebraic group, then $G \subset \mathrm{Aut}(D)^0$.*

Moreover, if $\mathrm{Aut}(D)^0$ *is conjugate to a subgroup of translations of dimension $n-2$, then* $\mathrm{Aut}(D) = \mathrm{Aut}(D)^0$ *is a connected algebraic group.*

Example 5.4. If $D : \Bbbk[u, v, x_1, \ldots, x_n] \to \Bbbk[u, v, x_1, \ldots, x_n]$ is a simple derivation as in Example 5.2, then $\mathrm{Aut}(D) = \Bbbk^n$ is a group of translations of maximal dimension.

In order to prove Theorem 5.3 we need a result which is interesting in its own right. Recall from Proposition 3.15 that we have a canonical regular action of $\mathrm{ACr}(n)$ on $\mathbb{A}^n$; then the simpleness of D implies that the restricted action is free:

Proposition 5.5. *If D is simple, then* $\mathrm{Aut}(D)$ *acts freely on* $\mathbb{A}^n_{\Bbbk}$.

Proof. By simplicity assume $\Bbbk = \mathbb{C}$, and denote $v = (g_1, \ldots, g_n)$ the vector field associated to D. Let φ be an automorphism in $\mathrm{Aut}(D)$ and $p \in \mathbb{A}^n_{\Bbbk}$ such that $\varphi(p) = p$; we need to prove that then $\varphi = \mathrm{Id}_{\mathbb{A}^n}$. Without loss of generality we may assume $p = (0, \ldots, 0)$.

Since v does not vanish at $(0, \ldots, 0)$, we need to add this result? or cite we have a unique (non-constant) local holomorphic solution of the differential equations system

$$x_i' = g(x_1, \ldots, x_n)\,,\ i = 1, \ldots, n,$$

that we denote $x_1 = x_1(t), \ldots, x_n = x_n(t)$, such that $x_1(0) = \cdots = x_n(0) = 0$. This solution defines a homomorphism of $\mathbb{C}$-algebras $\xi : \mathbb{C}[x_1, \ldots, x_n] \to \mathbb{C}[[t]]$, $x_i \mapsto x_i(t)$ for $i = 1, \ldots, n$, such that $\xi D = \partial_t \xi$ and $\xi^{-1}\big(t\mathbb{C}[[t]]\big) = \sum_{i=1}^n x_i \mathbb{C}[x_1, \ldots, x_n]$.

Now, the homomorphism $\xi_1 = \xi\varphi$ satisfies the same relations as ξ; set $y_i = \xi_1(x), i = 1, \ldots, n$. Then $\big(y_1(t), \ldots, y_n(t)\big)$ defines another solution of the differential equation above with the same initial conditions, so $\xi_1 = \xi$.

On the other hand, since $\ker\xi$ is invariant under D and contained in a maximal ideal we deduce that (recall that D is simple) $\ker\xi = 0$, i.e. ξ is injective. Thus $\varphi = \mathrm{Id}_{\mathbb{A}^n}$, as required. □

Proof of Theorem 5.3. (Sketch) First of all notice that $\mathrm{Aut}(D)^0$ inherits a structure of ind-variety from $\mathrm{Aut}(D)$, i.e., $\mathrm{Aut}(D)^0 = \bigcup_{r=0}^{\infty} \mathcal{B}_r$, where $\mathcal{B}_r = \mathcal{A}_r \cap \mathrm{Aut}(D)^0$ and therefore $\mathcal{B}_r \subset \mathcal{B}_{r+1}$ is a closed immersion of algebraic varieties for all $r \geq 1$. Since $\mathrm{Aut}(D)^0$ is connected one may suppose that $\mathcal{B}_r$ is connected for any r.[4] Notice that in this case, if $\mathcal{B}_r \subsetneq \mathcal{B}_{r+\ell}$ for some $\ell \geq 1$ such that $\dim \mathcal{B}_r = \dim \mathcal{B}_{r+\ell}$, and V is an irreducible component of $\mathcal{B}_{r+\ell}$ that is not contained in $\mathcal{B}_r$, then either $\mathcal{B}_r \cap V = \emptyset$ or it consists of singular points of $\mathcal{B}_{r+\ell}$.

Now take a point $p \in \mathbb{A}^n_{\Bbbk}$ and consider the "orbit" map $\mathrm{Aut}(D)^0 \to \mathbb{A}^n_{\Bbbk}$ given by $\varphi \mapsto \varphi(p)$. By Proposition 5.5, a fiber of that map intersects $\mathcal{B}_r$ in at most one point. Hence $\dim \mathcal{B}_r \leq n$ for all $r \geq 1$. Since $\mathrm{Aut}(D)^0$ acts (freely) on a singularity of any $\mathcal{B}_r$ by producing new singularities, we conclude it is necessarily contained in the set of non-singular points of one of them. In other words, $\mathrm{Aut}(D)^0 = \mathcal{B}_r$ for some r, which proves that it is an algebraic group of dimension at most n. Moreover, an element φ in that group admits a Jordan-Chevalley decomposition $\varphi_s\varphi_u$, where $\varphi_s, \varphi_u \in \mathrm{Aut}(D)^0$ are the corresponding semi-simple and unipotent parts. Since φ_s admits fixed points ([7, Prop. 15.9.3]) we deduce $\varphi_s = \mathrm{Id}_{\mathbb{A}^n}$. Thus $\mathrm{Aut}(D)^0$ is unipotent.

On the other hand, using the main results in [4] one can prove that $\dim \mathrm{Aut}(D)^0 \neq n, n-1$ from which we get assertion (a) (for more details see [16]).

To prove (b) we take $\varphi \in G$ and consider the closure $\overline{\langle\varphi\rangle}$ of the cyclic group generated by φ. Since it is an algebraic group, by arguing as above we one shows that $\varphi = \varphi_u$ is unipotent, hence there is a $\mathbb{G}_a$ containing φ. Thus $\mathbb{G}_a \subset \mathrm{Aut}(D)^0$ and we are done.

The proof of the last assertion is trickier and will be omitted ([16, Cor. 5.5.]). □

6. The functor of automorphisms — a glimpse on the schematic viewpoint

When presenting our problems, one of the main motivations was that $\mathrm{ACr}(n)$ cannot be endowed with a structure of algebraic group — or more generally, as group scheme. More precisely, we cannot endow $\mathrm{ACr}(n)$ with a structure of algebraic variety such that (a) the composition and the inverse are morphisms of algebraic varieties and (b) the canonical action of $\mathrm{ACr}(n)$ on $\mathbb{A}^n$ is regular. In this section we put this assertion into context. We need to assume that the reader has some background on algebraic geometry, that we will try to keep at the minimum requirements: we will rely on the results on ind-varieties presented in [7, Part 1, Chapters 1 and 2].

[4] Choose a connected component $\mathcal{B}'_1$ of $\mathcal{B}_1$, then a connected component $\mathcal{B}'_2$ of $\mathcal{B}_2$ containing $\mathcal{B}'_1$, and so on. Then rename and prove that $\bigcup_{r=1}^{\infty} \mathcal{B}'_r$ is open and closed in $\mathrm{Aut}(D)^0$.

6.1. **The automorphism group as an ind-variety.**

In Section 3 we showed how to endow $\mathrm{ACr}(n)$ with a topology (using the degree of an automorphism), in such a way that $\mathrm{ACr}(n)$ becomes an ind-variety. In this section we frame this construction in the context of the theory of algebraic varieties.

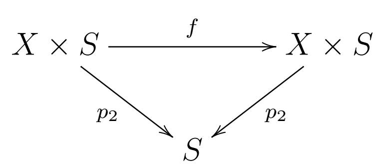
We first recall the following result on morphisms of ind-varieties:

Proposition 6.1. *Let X be an ind-variety, filtered as $X = \bigcup_{i\in\mathbb{N}} X_i$, and S an algebraic variety. If $f : S \to X$ is a continuous function, then there exists $j \in \mathbb{N}$ such that $f(S) \subset X_j$, In particular, $\overline{f(S)}$ is an algebraic variety, and if f is a morphism of ind-varieties, then $f : S \to X_j$ is a morphism of varieties.*

Proof. See [7, Lemma 1.1.5]. □

If $X = \mathrm{ACr}(n)$ and $f : S \to \mathrm{ACr}(n)$ is a morphism as in Proposition 6.1, we see f as a *continuous family of automorphisms* — since we have already a topology on $\mathrm{ACr}(n)$, this is reasonable. But assume for a moment that we *do not* have a topology on $\mathrm{ACr}(n)$. If this is the case, we can construct a topology using as departure point the fact that such an f "should be" a continuous family:

Definition 6.2. Let S, X be algebraic varieties. A *continuous family of automorphisms of $\mathbb{A}^n$ with parameter in S* is an automorphism of algebraic varieties $F : X \times S \to X \times S$ such that F commutes with the second projection $p_2 : X \times S \to S$



Thus, is $F : X \times S \to X \times S$ is a continuous family, then $F(x, s) = \big(f(x, s), s\big)$ for some morphism $f : S \times X \to X$. Moreover, since $F^{-1} : X \times S \to X \times S$ has the form $F^{-1}(x, s) = \big(g(x, s), s\big)$, we get that for a fixed $s \in S$, we have that $g\big(f(x, s), s\big) = \mathrm{Id}_{X\times S}$ and $f\big(g(x, s), s\big) = \mathrm{Id}_{X\times S}$ for all $x \in X$: $f(-, s) : X \to X$ and $g(-, s) : X \to X$ are morphisms of algebraic varieties, each being the inverse of the other.

We set a topology $\mathcal{T}_X$ on $\mathrm{Aut}(X)$ as follows: $\mathcal{T}_X$ is the finest topology (*i.e.* with more open subsets) on X such that the functions $s \mapsto f(-, s)$ are continuous for all continuous families $F : X \times S \to X \times S$. All we need to do now is to solve the following

Problem. *If $X = \mathrm{ACr}(n)$, study the relationship of the topology $\mathcal{T}_{\mathrm{ACr}(n)}$ and the topology given to $\mathrm{ACr}(n)$ as an ind-variety.*

Since the morphism $\mathbb{A}^n \times \mathcal{A}_r \to \mathbb{A}^n \times \mathcal{A}_r$, given by $(v, f) \mapsto \big(f(v), f\big)$ is an automorphism, we deduce that the inclusion $\mathcal{A}_r \hookrightarrow \mathrm{ACr}(n)$ is a continuous family. Therefore, is $Z \subset \mathrm{ACr}(n)$ is closed in the $\mathcal{T}_{\mathrm{ACr}}(n)$ topology, then $Z \cap \mathcal{A}_r \subset \mathcal{A}_r$ is closed. So, $\mathcal{T}_X$ is finer that $\mathcal{A}_r$ if and only if the following question has positive answer.

Question. Is $\mathcal{A}_r$ closed in the topology $\mathcal{T}_{\mathrm{ACr}(n)}$?

Since we are looking for a geometric description, from the categorical viewpoint, to restrain ourselves to the topology does not give the full picture. In order to have a better understanding, we need to take a more categorical approach.

6.2. **The functor of automorphisms.**

The construction of continuous families have "functorial properties":

Let $\mathcal{AV}$ be the category of algebraic $\Bbbk$-varieties, and denote by $\mathcal{AV}^{op}$ its opposite category (that is, we reverse the arrows).

Exercise 6.3. The assignments

$$S \mapsto \big\{F : X \times S \to X \times S : p_2 F = p_2 \text{ and } \exists F^{-1}\big\}$$
$$(g : S' \to S) \mapsto \big(F \mapsto F \circ (\mathrm{Id}_X, g)\big)$$

provide a functor $\mathcal{AV} \to \mathit{Groups}$, that we will denote as Aut_X.

Remembering Yoneda's Lemma[5], we can rephrase our "lack of algebraic variety structure" of $\mathrm{ACr}(n)$ result as follows:

Proposition 6.4. *The functor $\mathit{Aut}_{\mathbb{A}^n}$ is not representable by an algebraic variety — that is, if X is an algebraic variety, then* $\mathrm{Hom}(-, X) \neq \mathit{Aut}_{\mathbb{A}^n}$. □

6.3. **The schematic viewpoint.**

In fact, the constructions of the preceding section not only hold when we work with schemes[6], but from the perspective of modern algebraic geometry, become more meaningful in the schematic setting:

Definition 6.5. Let Sch be the category of $\Bbbk$-schemes and X, Y be $\Bbbk$-schemes. We define the functor $\mathit{Hom}_{X,Y} : \mathit{Sch}^{op} \to \mathit{Sets}$ by

$$S \mapsto \mathit{Hom}_{X,Y}(S) = \big\{F : X \times S \to Y \times S : p_2 F = p_2 \text{ and } \exists F^{-1}\big\}$$
$$(g : S' \to S) \mapsto \big(F \mapsto F \circ (\mathrm{Id}_X, g)\big)$$

The set $\mathit{Hom}_{X,Y}(S)$ is called the set of *S-points* of the functor. If $X = Y$ we set $\mathit{End}_X = \mathit{Hom}_{X,Y}$. We denote by $\mathit{Aut}_X \subset \mathit{End}_X$ the subfunctor of automorphisms $F : X \times S \to X \times S$.

Exercise 6.6. If X is a scheme, prove that End_X can be seen as a functor on monoids, and that Aut_X can be seen as a functor on groups.

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[5]Roughly speaking, this result from the theory of categories states that is $\mathcal{C}$ is a small category and $X_i \in \mathcal{C}$, then the functors $\mathrm{Hom}(-, X_i) : \mathcal{C}^{op} \to \mathit{Sets}$ are naturally equivalent if and only if $X_1 \cong X_2$ — in other words, we can distinguish objects in $\mathcal{C}$ using the functor $\mathrm{Hom}(-, X)$.

[6]In fact, the definition of the functor Aut_X is already in Grothendieck's EGA.

Remark 6.7. (1) Under this point of view, we have that $\text{Hom}(X, Y)$ is the set of $\Bbbk$-points of the functor $Hom_{X,Y}$. Analogously,

$$\text{ACr}(n) \cong Aut_{\mathbb{A}^n}(\Bbbk)$$

(2) Notice that Yoneda's Lemma states that if $Hom_{X,Y}$ is represented by a scheme Z, then an element $F \in Hom_{X,Y}(S)$ corresponds to a *morphism of schemes* $\sigma : S \to Z$, given by $\sigma(s) = \big(x \mapsto f(x, s)\big)$ if $F(x, s) = \big(f(x, s), s\big)$.

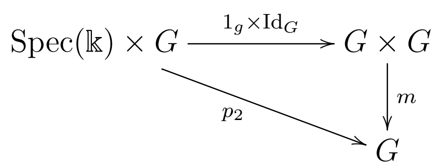
(3) Thus, under this point of view the construction of the topology $\mathcal{T}_{\text{ACr}}(n)$ would be the first step to take in order to represent the functor $Aut_{\mathbb{A}^n}$ by a scheme — since for example any morphism of schemes is continuous —, a goal that we know cannot be attained:

Definition 6.8. A *scheme on groups* or *group scheme* is a quadruple $(G, m, \iota, 1_G)$ where $m : G \times G \to G$ is an associative morphism of schemes, and $\iota, 1_G$ are morphisms of schemes, with $1_G, \iota, m$ satisfying the commutative diagrams associated to the fact that G is a group.

Remark 6.9. (1) $\text{Spec}(\Bbbk)$ is the scheme that has a unique point as base topological space, with $\Bbbk$ as the algebra of regular functions.

(2) Notice that the neutral element has to be defined as a $\text{Spec}(\Bbbk)$-point of G.

(3) For example, the fact that 1_G is a neutral element at left is expressed as the commutativity of the following diagram:



(4) A group scheme can be defined as a representable functor $G : Sch^{op} \to Groups$.

The we have the following result, which proof uses some basic results on the description of schemes as functors.

Theorem 6.10. *The functor $Aut_{\mathbb{A}^n}$ is not represented by a scheme.* □

Since we know that $\text{ACr}(n)$ admits a structure of ind-variety, the following problem arises

Problem. *Is the functor $Aut_{\mathbb{A}^n}$ represented by an ind-scheme?*

It should be noted that whereas the notion of *ind-scheme* at first view seems similar to the one of ind-variety[7], its study involves a deep understanding of the theory of schemes (in particular, formal schemes).

[7] An *ind-scheme* is a functor $\mathit{AffSch}^{op} \to \mathit{Sets}$ represented by an inductive limit of schemes.

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