MASTER CLASS: REAL FORMS OF TORIC VARIETIES

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Abstract. Toric varieties are central objects in algebraic geometry, whose combinatorial description makes them especially tractable. In this mini-course, we will first review their classification over an algebraically closed field. We will then explain how Galois descent can be used to obtain a classification over the field of real numbers.

Contents

1. Toric varieties (Lecture 1)

We work over an algebraically closed field K of characteristic 0 (e.g. $k = \mathbb{C}$). Our main reference for the preparation of these lectures is the first chapter of the book *Introduction to Toric Varieties* by William Fulton.

Recall that an algebraic variety is a geometric object defined locally by polynomial equations. For example:

- The affine space $\mathbb{A}^n$, or the curves in $\mathbb{A}^2$ with affine coordinates (x, y), such as those defined by $Z(xy - 1)$ or $Z(x^2 + y^2 - 1)$ (which are actually isomorphic over K, do you see why?);
- The projective space $\mathbb{P}^n = (K^{n+1} \setminus \{0\})/K^*$, or a projective curve in $\mathbb{P}^3$ with homogeneous coordinates $[x : y : z : w]$, defined by $Z(xz-y^2,\ yw-z^2,\ xw-yz)$.

An *algebraic group* is an algebraic variety endowed with a group structure such that the multiplication and the inverse maps

$$m : G \times G \to G \quad \text{and} \quad \iota : G \to G$$

are morphisms of algebraic varieties (i.e. given by polynomial maps in coordinates). For instance, GL_n, SL_n, O_n, SO_n, Sp_{2n}, etc., are algebraic groups (but there are many more examples). We define the multiplicative group, denoted $\mathbb{G}_{\mathrm{m}}$, as

$$\mathbb{G}_{\mathrm{m}} = (K^*, \cdot),$$

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viewed as the affine curve $Z(xy-1) \subset \mathbb{A}^2$. An algebraic torus of dimension n (with $n \geq 0$) is the algebraic group

$$T = \mathbb{G}_{\mathrm{m}}^n.$$

An algebraic variety U is called a T-variety if there is a faithful algebraic action of T on U, i.e. the action map

$$T \times U \to U, (t,u) \mapsto t \cdot u$$

is a morphism of algebraic varieties and $K := \{t \in T \mid \forall u \in U,\ t \cdot u = u\} = \{1\}$.

Definition 1.1. Let U be a T-variety. Assume that U is normal (i.e. its local rings are integrally closed domains), and that T acts on U with a dense open orbit (for the Zariski topology). Then U is called a toric variety.

- A first elementary example: $\mathbb{A}^1$, with the action of $\mathbb{G}_m$ given by multiplication $t \cdot x = tx$, and $\mathbb{P}^1$, with the standard torus action $t \cdot [x_0 : x_1] = [x_0 : tx_1]$ extending that on $\mathbb{A}^1$.
- A two-dimensional example is the surface

  $$S = Z(v^2 - uw) \subset \mathbb{A}^3.$$

  The torus $T = (\mathbb{C}^*)^2$ acts on S by

  $$(t_1, t_2) \cdot (u, v, w) = (t_1 u,\ t_1 t_2 v,\ t_1 t_2^2 w),$$

  which preserves the equation since both v^2 and uw scale by $t_1^2 t_2^2$. The dense open orbit is

  $$O = T \cdot (1,1,1) \simeq (\mathbb{C}^*)^2.$$

  There are moreover two 1-dimensional orbits (do you see which ones?) and a fixed point $\{(0,0,0)\}$.

Toric varieties are important because they provide an elementary and highly explicit setting in which many constructions and phenomena in algebraic geometry can be studied. For instance, they offer a concrete framework for cohomology theories, resolution of singularities, Hodge theory, mirror symmetry, intersection theory, Riemann–Roch theorems, vanishing theorems, the Manin conjecture, and the minimal model program, among others.

Let us now explain, through an example, how to construct a toric variety from a strongly convex polyhedral cone σ in a lattice $N \simeq \mathbb{Z}^n$.

Example 1.2. Let $N = \mathbb{Z}^2$ with basis $\{e_1, e_2\}$ and dual basis $\{e_1^*, e_2^*\}$. Consider the cone

$$\sigma = \mathrm{Cone}(e_2,\, 2e_1 - e_2) \subset N_{\mathbb{R}} = \mathbb{R}^2.$$

The dual cone

$$\sigma^\vee = \{\varphi \in M_{\mathbb{R}} \mid \forall v \in \sigma,\ \langle \varphi, v \rangle \geq 0\}, \quad M = \mathrm{Hom}(N, \mathbb{Z}),$$

is determined by

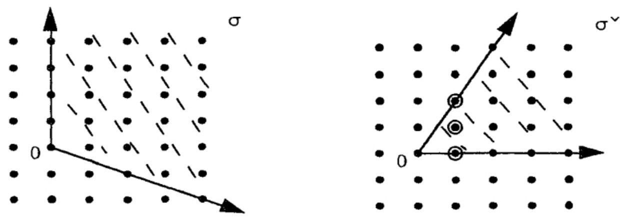
$$\langle \varphi, e_2 \rangle \geq 0, \qquad \langle \varphi, 2e_1 - e_2 \rangle \geq 0.$$

Writing $\varphi = ae_1^* + be_2^*$, we obtain

$$b \geq 0, \qquad 2a - b \geq 0,$$

hence

$$\sigma^\vee = \mathrm{Cone}(e_1^*,\, e_1^* + 2e_2^*).$$



The corresponding semigroup is

$$S_\sigma = \sigma^\vee \cap M.$$

It is generated by

$$e_1^*, \quad e_1^* + e_2^*, \quad e_1^* + 2e_2^*.$$

Introduce variables

$$u = \chi^{e_1^*}, \quad v = \chi^{e_1^*+e_2^*}, \quad w = \chi^{e_1^*+2e_2^*}.$$

Then the semigroup algebra is

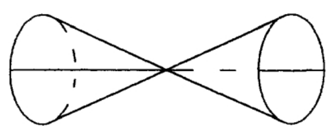
$$\mathbb{C}[S_\sigma] = \mathbb{C}[u,v,w]/(v^2 - uw),$$

since the only relation (up to multiple) among generators is

$$2(e_1^* + e_2^*) = e_1^* + (e_1^* + 2e_2^*).$$

We therefore associate to the cone σ the affine variety

$$U_\sigma := Z(v^2 - uw) \subset \mathbb{A}^3.$$



It turns out that this recipe works in full generality. More precisely, we have the following result.

Theorem 1.3 (Affine toric correspondence)**.** *Let $N \simeq \mathbb{Z}^n$ and let $T = \mathbb{G}_m^n$. There is a one-to-one correspondence (up to isomorphism) between:*

- *strongly convex rational polyhedral cones $\sigma \subset N_\mathbb{R}$; and*
- *affine toric varieties U_σ with acting torus T.*

More precisely, to a cone $\sigma \subset N_\mathbb{R}$ one associates the affine toric variety

$$U_\sigma := \operatorname{Spec}(\mathbb{C}[S_\sigma]),$$

and every affine toric variety arises uniquely in this way up to isomorphism.

In general, a toric variety need not be affine; nevertheless, it can be covered by affine toric varieties with the same acting torus T by Sumihiro's theorem (1974). Thus, any toric variety is obtained by gluing affine toric varieties along dense open subsets. This gluing is encoded combinatorially by the notion of a fan that we now define.

Definition 1.4. A *fan* Σ in $N \simeq \mathbb{Z}^n$ is a finite collection of strongly convex rational polyhedral cones $\sigma \subset N_\mathbb{R}$ such that:

- every face of a cone in Σ also belongs to Σ;

- the intersection of any two cones in Σ is a face of each.

Given a fan Σ, one constructs the toric variety $X(\Sigma)$ by gluing the affine toric varieties U_σ, for $\sigma \in \Sigma$, along their common open subsets. Indeed, for cones $\sigma, \tau \in \Sigma$, the intersection $\sigma \cap \tau$ is a face of both, so $U_{\sigma\cap\tau}$ identifies with a dense open subset of both U_σ and U_τ. Gluing along these open subsets yields a toric variety. Moreover, every toric variety arises in this way.

Theorem 1.5 (Toric correspondence)**.** *Let $N \simeq \mathbb{Z}^n$ and let $T = \mathbb{G}_m^n$. There is a one-to-one correspondence (up to isomorphism) between: There is a one-to-one correspondence between:*
- *fans Σ in $N_\mathbb{R}$; and*
- *toric varieties $X(\Sigma)$ with acting torus T.*

More precisely, to a fan Σ one associates the toric variety $X(\Sigma)$ and every toric variety arises uniquely in this way up to isomorphism.

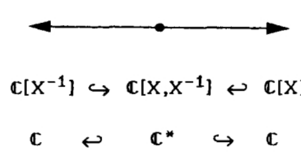
Example 1.6 (Classification of toric varieties in dimension 1)**.** Let $N \simeq \mathbb{Z}$, so $N_\mathbb{R} \simeq \mathbb{R}$. A fan in $N_\mathbb{R}$ is a finite collection of cones whose support is contained in $\mathbb{R}$ and whose cones are faces of each other. Hence every fan in dimension 1 is of the form

$$\Sigma = \{\{0\}, \sigma_1, \dots, \sigma_k\},$$

where each σ_i is a strongly convex cone, hence either $\mathbb{R}_{\geq 0}$ or $\mathbb{R}_{\leq 0}$, and the fan condition forces that we can have at most these two rays.

Up to isomorphism, there are only three possibilities:

- **The trivial fan:** $\Sigma = \{\{0\}\}$. The associated toric variety is $X_\Sigma \simeq \mathbb{G}_m$.
- **The fan with one ray:** $\Sigma = \{\{0\}, \mathbb{R}_{\geq 0}\}$ (or equivalently $\mathbb{R}_{\leq 0}$). The associated toric variety is $X_\Sigma \simeq \mathbb{A}^1$.
- **The complete fan with two rays:** $\Sigma = \{\{0\}, \mathbb{R}_{\geq 0}, \mathbb{R}_{\leq 0}\}$.



The associated toric variety is obtained by gluing two copies of $\mathbb{A}^1$ along $\mathbb{A}^1 \setminus \{0\}$, hence $X_\Sigma \simeq \mathbb{P}^1$.

Therefore, up to isomorphism, the only toric varieties of dimension 1 are:

$$\mathbb{G}_m, \quad \mathbb{A}^1, \quad \mathbb{P}^1.$$

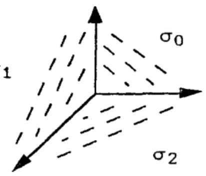
Example 1.7 (The toric surfaces $\mathbb{P}^2$ and $\mathbb{F}_a$)**.** Let $N = \mathbb{Z}^2$ with standard basis e_1, e_2 and $N_\mathbb{R} = \mathbb{R}^2$.

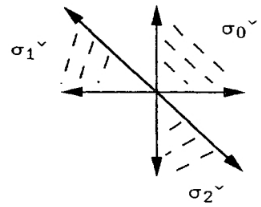
The projective plane $\mathbb{P}^2$. Consider the fan whose rays are generated by

$$v_1 = e_1, \qquad v_2 = e_2, \qquad v_3 = -e_1 - e_2.$$

Its maximal cones are

$$\sigma_0 = \langle v_1, v_2 \rangle, \qquad \sigma_1 = \langle v_2, v_3 \rangle, \qquad \sigma_2 = \langle v_3, v_1 \rangle.$$





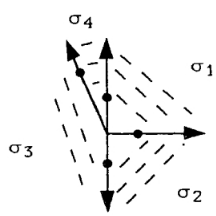
Each cone corresponds to an affine toric surface isomorphic to $\mathbb{A}^2$, and gluing along the overlaps determined by the fan yields $X_\Sigma \simeq \mathbb{P}^2$.

The Hirzebruch surface $\mathbb{F}_a$ ($a \geq 0$). Consider the fan whose rays are generated by

$$v_1 = e_1, \qquad v_2 = e_2, \qquad v_3 = -e_2, \qquad v_4 = -e_1 + ae_2.$$

Its maximal cones are

$$\sigma_1 = \langle v_1, v_2 \rangle, \qquad \sigma_2 = \langle v_1, v_3 \rangle, \qquad \sigma_3 = \langle v_3, v_4 \rangle, \qquad \sigma_4 = \langle v_4, v_2 \rangle.$$



Each cone corresponds to an affine toric surface isomorphic to $\mathbb{A}^2$, and gluing along the overlaps determined by the fan yields the smooth complete toric surface

$$X_\Sigma \simeq \mathbb{F}_a \simeq \mathbb{P}(\mathcal{O}_{\mathbb{P}^1} \oplus \mathcal{O}_{\mathbb{P}^1}(a))\,.$$

Note that $\mathbb{F}_1$ is the blow-up of $\mathbb{P}^2$ at the point $[1:0:0]$.

To finish this first lecture, let us give two examples illustrating how the combinatorial point of view simplifies the study of geometric properties of toric varieties.

Proposition 1.8. *Let X_Σ be a toric variety associated with a fan Σ in $N_\mathbb{R}$.*

(i) *The toric variety X_Σ is complete if and only if*

$$\bigcup_{\sigma \in \Sigma} \sigma = N_\mathbb{R}.$$

(ii) *The toric variety X_Σ is smooth if and only if every cone $\sigma \in \Sigma$ is generated by part of a basis of the lattice N.*

As mentioned above, the main interest of the theory of toric varieties is that essentially all geometric data can be computed on the combinatorial side. It is therefore natural to ask whether the theory can be extended from algebraically closed fields of characteristic zero to more general fields, or at least to arbitrary fields of characteristic zero (such as the field $\mathbb{R}$). This will be addressed in the next two lectures.

2. Real forms (Lecture 2)

Let X be a complex variety given by real polynomials. Then we may consider its points in $\mathbb{R}^n$ instead of $\mathbb{C}^n$. Naively, this is what one calls a *real variety.* However, one should be careful with the fact that a real variety is defined by its polynomials rather than by its solutions in $\mathbb{R}^n$. Examples will help to make this point clear.

Example 2.1 (The circle)**.** Consider the variety $\mathbb{S}^1$ given by the equation $x^2+y^2=1$. Its solutions in $\mathbb{R}^2$ define a circle, which a real variety. Its complex points however behave rather like a line minus a point, i.e. like $\mathbb{C}^*$. Indeed, the (complex) change of variables $z=x+iy$ and $w=x-iy$ tells us that, as a complex variety, this is also given by the polynomial $zw=1$, which is isomorphic via the projection on the first variable to the complex line $\mathbb{C}$ minus 0, that is, $\mathbb{C}^*$.

Example 2.2 (The multiplicative group)**.** Consider now the variety $\mathbb{G}_{\mathrm{m},\mathbb{R}}$ given by the equation $xy=1$. Its solutions in $\mathbb{R}^2$ are isomorphic to $\mathbb{R}^*$ via the projection on the first variable. Of course, this is also the case with $\mathbb{C}^*$ if we focus on this equation as a complex variety.

Example 2.3 (The pointless (affine) conic)**.** Consider the variety C given by the equation $x^2+y^2=-1$. It has no solutions in $\mathbb{R}^2$, so one would think that there is no interesting structure to be studied here. However, its complex points are still interesting. Via the complex change of variables $z=ix-y$ and $w=ix+y$ we get that this variety is also given by the polynomial $zw=1$, so we recover once again $\mathbb{C}^*$, but starting with a different object over the real numbers.

One can analogously consider the two-dimensional real variety Q given by $x^2+y^2+z^2=-1$. As a subset of $\mathbb{R}^3$, it has again no points at all. But looking at its complex points we realize that this deserves indeed to be called a two-dimensional variety and that it must represent then a different object than C, even though they both have an empty set of real points.

This last example tells us that the focus should be more on the polynomial equations defining the varieties than on the points themselves if we are to work over the real numbers. But it also tells us that a good idea is to look at its complex points and somehow keep track of the fact that these solutions come from real polynomials. This can be done by looking at fixed points by the action of complex conjugation.

Denote by $\Gamma=\{1,s\}\simeq\mathbb{Z}/2\mathbb{Z}$ the Galois group of $\mathbb{C}/\mathbb{R}$, so that s is the complex conjugation. Then $\mathbb{R}^n$ can be recovered as the fixed point set $(\mathbb{C}^n)^s$. If X is a complex variety defined by real polynomials, any point $x\in X$ must be mapped to another point in X via s. In other words, s acts on X. Then the real points of X, which we will denote by $X(\mathbb{R})$, are nothing but the fixed points X^s. Note that his action is via an anti-holomorphic automorphism of X.

Let us see if we can tell the difference between the three examples above, which all correspond to the complex variety $X=\mathbb{C}^*$, with this data.

In the first case, the real points $\mathbb{S}^1(\mathbb{R})$ of the circle are obtained as solutions to the equation $x^2+y^2=1$ in $\mathbb{C}^2$ that are in $\mathbb{R}^2$. That is, they are stable under the action of s that sends (x,y) to $(\bar{x},\bar{y})$. If we push this via the change of variables $z=x+iy$ and $w=x-iy$ that allowed us to view this circle as $\mathbb{C}^*$ as a complex variety, then we see that the action of s sends (z,w) to $(\bar{w},\bar{z})$, which inside X can be rewritten as $(\bar{z}^{-1}, \bar{w}^{-1})$. After projection on the first variable, we see then that we have $\mathbb{C}^*$ with an action of s given by $z \mapsto \bar{z}^{-1}$.

In the second case, since the equation over the real numbers is $xy = 1$, there is no change of variables whatsoever. We see then that the action of s on the points of X is the usual one, by complex conjugation on each variable. After projection, we see then that we have just recovered $\mathbb{R}^*$ as the fixed points of $\mathbb{C}^*$ via the usual complex conjugation.

Finally, in the third case, we may push the action of s via the change of variables $z = ix - y$ and $w = ix + y$. We recover then an action on the complex variety X, with equation $zw = 1$, given by $(z, w) \mapsto (-\bar{w}, -\bar{z})$, which can be rewritten as $(-\bar{z}^{-1}, -\bar{w}^{-1})$. After projection on the first variable, we see then that this is the action on $\mathbb{C}^*$ given by $z \mapsto -\bar{z}^{-1}$, which has no fixed points.

The fact that we are able to somehow recover all the information from the action of s justifies the following definition.

Definition 2.4. Let X be a complex variety. We define a *real form* of X as a pair (X, a), where a is an anti-holomorphic automorphism of X.

One can prove that this is equivalent to the data of a complex variety locally defined by real polynomials (both for the affine pieces and the gluing data). Then the fixed points X^a correspond to the real solutions of these polynomials.

Example 2.5 (Forms of $\mathbb{P}^1_{\mathbb{C}}$)**.** The projective line $\mathbb{P}^1_{\mathbb{C}}$ is obtained as the gluing of two copies of $\mathbb{A}^1_{\mathbb{C}}$ by identifying the open subsets $\mathbb{G}_{\mathrm{m},\mathbb{C}} \subseteq \mathbb{A}^1_{\mathbb{C}}$ via $x \mapsto x^{-1}$. Since this is a real polynomial, this gluing works over $\mathbb{R}$ and defines the projective line $\mathbb{P}^1_{\mathbb{R}}$. But we can also obtain this variety by considering the natural action of s on both copies of $\mathbb{A}^1_{\mathbb{C}}$, which is compatible with the gluing (since $\overline{(z^{-1})} = (\bar{z})^{-1}$). In terms of homogeneous coordinates, this corresponds to

$$[x : y] \mapsto [\bar{x} : \bar{y}].$$

Then $\mathbb{P}^1_{\mathbb{R}}$ corresponds to $\mathbb{P}^1_{\mathbb{C}}$ equipped with this action.

One can consider another antiholomorphic action that, besides applying complex conjugation, permutes the two copies of $\mathbb{A}^1_{\mathbb{C}}$. This defines an action on the intersection, that is isomorphic to $\mathbb{G}_{\mathrm{m},\mathbb{C}}$. We may force this action to be the one defining the "pointless" conic C. In terms of homogeneous coordinates, this reads

$$[x : y] \mapsto [-\bar{y} : \bar{x}].$$

We obtain then a projective "pointless" conic $\bar{C}$ that contains C as an affine open subset. In terms of homogeneous polynomials $[x : y : z]$ in $\mathbb{P}^2_{\mathbb{R}}$, it corresponds to $x^2 + y^2 + z^2 = 0$.

Given these examples, one could wonder whether there is a systematic way of doing this for a general complex variety X. More precisely, there are two natural questions for such an X:

(i) Does X admit a real form?
(ii) If this is the case, how many real forms are there up to isomorphism?

The goal of the next sections will be to present the tools that are necessary to answer the second question, and give a precise criterion for a positive answer of the first question in the context of toric varieties.

Recalling that we are interested in toric varieties, we need an analogous treatment for algebraic groups (particularly for tori) and more generally for varieties equipped with a group action. Let us extend then this notion to the context of algebraic group first, and we will leave toric varieties for after we introduce the tools of Galois cohomology and its relation with torsors.

Definition 2.6. Let G be a complex algebraic group. We define a *real form* of G as a real form of (G, a) of the variety G such that a preserves the group structure.

Note that in particular we must have $a(e) = e$, where $e \in G$ denotes the neutral element. This tells us that a real form of an algebraic group always has real points.

Example 2.7. The multiplicative group $\mathbb{G}_{\mathrm{m},\mathbb{C}}$, which is the variety $\mathbb{C}^*$ equipped with its usual complex multiplication, is an algebraic group. The usual complex conjugation is clearly a group automorphism, so that the real variety given by the equation $xy = 1$ and whose real points are $\mathbb{R}^*$ is a real form of this group, which we denote by $\mathbb{G}_{\mathrm{m},\mathbb{R}}$.

Example 2.8. There is another form of the multiplicative group $\mathbb{G}_{\mathrm{m},\mathbb{C}}$. Indeed, the anti-holomorphic automorphism $z \mapsto \bar{z}^{-1}$ is easily seen to be a group homomorphism as well. Since this action defines the circle $\mathbb{S}^1$, we see that $\mathbb{S}^1$ naturally inherits a group structure. And it is indeed the usual complex multiplication restricted to the circle.

In terms of solutions to the equation $x^2 + y^2 = 1$, this reads

$$(x_1, y_1)(x_2, y_2) = (x_1x_2 - y_1y_2, x_1y_2 + x_2y_1).$$

Example 2.9. Let us consider now the complex torus $\mathbb{G}_m^2 = (\mathbb{C}^*)^2$, equipped with the following anti-holomorphic automorphism:

$$a : (z, w) \mapsto (\bar{w}, \bar{z}).$$

Since complex conjugation respects multiplication and the group structure is defined coordinate-wise, we immediately see that this is a group automorphism, and hence the pair $(\mathbb{G}_m^2, a)$ defines a real form of $\mathbb{G}_{\mathrm{m},\mathbb{C}}^2$ denoted as $R_{\mathbb{C}/\mathbb{R}}(\mathbb{G}_{\mathrm{m},\mathbb{C}})$. It receives the name of "Weil Restriction of scalars (from $\mathbb{C}$ to $\mathbb{R}$) of $\mathbb{G}_{\mathrm{m},\mathbb{C}}$" and it corresponds to the vague idea of "looking at the points of $\mathbb{G}_{\mathrm{m},\mathbb{C}} = \mathbb{C}^*$ as the $\mathbb{R}$-points of a real algebraic group. In fact, a direct computation tells us that

$$\begin{aligned} R_{\mathbb{C}/\mathbb{R}}(\mathbb{G}_{\mathrm{m},\mathbb{C}})(\mathbb{R}) &= \{(z, w) \in (\mathbb{C}^*)^2 \mid (z, w) = (\bar{w}, \bar{z})\} \\ &= \{(z, w) \in (\mathbb{C}^*)^2 \mid z = \bar{w}\} \\ &\simeq \{z \in \mathbb{C}^*\} = \mathbb{C}^*. \end{aligned}$$

3. Galois cohomology (Lecture 2)

For simplicity, all fields are assumed to be of characteristic 0.

Let L/K be a finite Galois extension with Galois group Γ. For all practical purposes afterwards, one may assume that L/K is simply $\mathbb{C}/\mathbb{R}$, so that $\Gamma \simeq \mathbb{Z}/2\mathbb{Z}$ and its unique nontrivial element is complex conjugation.

When we defined real forms, we obtained them by "going down" from $\mathbb{C}$ to $\mathbb{R}$ by taking fixed points with respect to an anti-holomorphic involution. But we also mentioned that this corresponds to real polynomials, which define real varieties that look like the complex variety we started with when we look at complex solutions (i.e. "going up"). This is a better point of view for our more general context.

Definition 3.1. A K-variety X is a K-form of an L-variety Y if we have an isomorphism of L-varieties $Y \simeq X_L$, where X_L denotes the base change from K to L (i.e. same polynomial equations, but seen inside L^n instead of K^n).

Analogously, we say that a K-group G is a K-form of an L-group H if we have an isomorphism of L-groups $H \simeq G_L$.

Remark 3.2. This definition can be extended to essentially any category of algebraic objects that have a good notion of "being defined over L" and a nice "group of L-automorphisms" for any finite extension L/K. This will be important when defining and classifying real torsors, and more generally real toric varieties, further below.

The two questions we naturally asked for real forms can be extended to this context. Given an L-variety Y:
(i) Does Y admit a K-form?
(ii) If this is the case, how many K-forms are there up to isomorphism?
In this section, we will not deal with the first question, since it is already hard enough when $K = \mathbb{R}$. Note however that, in the context of toric varieties, one can actually give an explicit criterion for this, which is even easier when $K = \mathbb{R}$.

Assume then that the answer to the first question is yes or, which amounts to the same, suppose that we start with a K-variety X and we want to classify all the K-forms of X_L. There is of course X itself, but there could be a lot more of them. In order to do this, consider the following two groups that act naturally on X_L:
- The group $A = \mathrm{Aut}_L(X_L)$ of L-automorphisms of the L-variety X_L.
- The Galois group Γ. In this last case, the action comes from the natural action of Γ on L, and hence on L^n, where X_L sits. This is *not* an action by automorphisms of L-varieties (but they are automorphisms of K-schemes).

These two actions talk to each other. One can actually define an action of Γ on A by conjugation. If we denote by σ_s the automorphism of X_L defined by $s \in \Gamma$, then for $\varphi \in \mathrm{Aut}_L(X_L)$ we define

$$ {}^s\varphi := \sigma_s \circ \varphi \circ \sigma_s^{-1}. $$

One can check that this is indeed an automorphism of L-varieties, hence an element in $\mathrm{Aut}_L(X_L)$, even though σ_s is not.

Let us consider now a K-form Y of X_L. This means that there exists an isomorphism of L-varieties $f : X_L \to Y_L$. On the other hand, the group Γ acts on both X_L and Y_L as it was defined above. We will denote these actions by σ^X and σ^Y respectively. One can check then that the composition

$$ a_s := f^{-1} \circ \sigma_s^Y \circ f \circ (\sigma_s^X)^{-1}, $$

is an automorphism of X_L and actually corresponds to an element in $\mathrm{Aut}_L(X_L)$. Then a direct computation gives that, for $s, t \in \Gamma$,

$$ \begin{aligned} a_{st} &= f^{-1} \circ \sigma_{st}^Y \circ f \circ (\sigma_{st}^X)^{-1} \\ &= f^{-1} \circ \sigma_s^Y \sigma_t^Y \circ f \circ (\sigma_t^X)^{-1} (\sigma_s^X)^{-1} \\ &= f^{-1} \circ \sigma_s^Y \circ (f \circ (\sigma_s^X)^{-1} \circ \sigma_s^X \circ f^{-1}) \circ \sigma_t^Y \circ f \circ (\sigma_t^X)^{-1} (\sigma_s^X)^{-1} \\ &= (f^{-1} \circ \sigma_s^Y \circ f \circ (\sigma_s^X)^{-1}) \circ \sigma_s^X \circ (f^{-1} \circ \sigma_t^Y \circ f \circ (\sigma_t^X)^{-1})(\sigma_s^X)^{-1} \\ &= a_s \circ {}^s a_t \end{aligned} $$

Definition 3.3. A function $a : \Gamma \to A$ satisfying the relation $a_{st} = a_s \circ {}^s a_t$ for every $s, t \in \Gamma$ is what is called a 1-cocycle. The set of such cocycles is denoted as $Z^1(\Gamma, A)$.

Assume now that we started with another isomorphism $f' : X_L \to Y_L$, yielding the cocycle $b_s = {f'}^{-1} \circ \sigma_s^Y \circ f' \circ (\sigma_s^X)^{-1}$. Then, defining $\varphi := {f'}^{-1} \circ f \in \mathrm{Aut}_L(X_L)$, a direct computation gives, for any $s \in \Gamma$.

$$\begin{aligned} b_s &= {f'}^{-1} \circ \sigma_s^Y \circ f' \circ (\sigma_s^X)^{-1} \\ &= {f'}^{-1} \circ (f \circ f^{-1}) \circ \sigma_s^Y \circ (f \circ (\sigma_s^X)^{-1} \circ \sigma_s^X \circ f^{-1}) \circ f' \circ (\sigma_s^X)^{-1} \\ &= ({f'}^{-1} \circ f) \circ (f^{-1} \circ \sigma_s^Y \circ f \circ (\sigma_s^X)^{-1}) \circ \sigma_s^X \circ (f^{-1} \circ f') \circ (\sigma_s^X)^{-1} \\ &= \varphi \circ a_s \circ {}^s\varphi^{-1} \end{aligned}$$

Definition 3.4. We define $H^1(\Gamma, A)$ as the quotient of $Z^1(\Gamma, A)$ by the equivalence relation given by $a \sim b$ if and only if there exists $\varphi \in A$ such that $b_s = \varphi \circ a_s \circ {}^s\varphi^{-1}$ for every $s \in \Gamma$. This is known as the *first Galois cohomology set* with values in the Γ-group A.

Remark 3.5. As the last sentence suggests, this definition makes sense for any Γ-group, that is, any group A (with group operation denoted by $\circ$) equipped with a Γ-action by group automorphisms.

If the group A is abelian, then both $Z^1(\Gamma, A)$ and $H^1(\Gamma, A)$ have natural abelian group structures, but in general there is no group structure on this set. There is however a notion of "trivial" element, which is the trivial function $\mathbf{1} : \Gamma \to A$ that sends every $s \in \Gamma$ to the identity id_{X_L} in A.

We have thus seen a way to explicitly construct a map

$$\Phi : \{K\text{-forms of } X_L\} \to H^1(\Gamma, A).$$

In this context, the obvious K-form X we started with corresponds precisely to the trivial element mentioned in the remark above. We have then the following result.

Theorem 3.6. *The map Φ described above is a bijection. In particular, K-forms of the L-variety X_L are classified by the set $H^1(\Gamma, A)$.*

The proof of this result requires the tools of Galois descent, which are far from the scope of this mini-course. Let us simply remark that, given a cocycle $a \in Z^1(\Gamma, A)$, one can recover an action of Γ on X_L via $a_s \circ \sigma_s$. The corresponding K-form of X_L is obtained by "taking invariants" via this action (which, again, is not an action by automorphisms of L-varieties). Equivalent cocycles will then give rise to (K-)isomorphic K-forms.

Remark 3.7. The result is also true for K-forms of other algebraic objects (see Remark 3.2). In particular, when G an algebraic K-*group* and A is the group of L-*group* automorphisms $\mathrm{Aut}_{L-\mathrm{gp}}(G_L)$, $H^1(\Gamma, A)$ classifies K-forms of the L-group G_L.

Example 3.8. Assume that L/K is $\mathbb{C}/\mathbb{R}$, as in the previous section. Then $\Gamma = \{1, s\}$ and a cocycle is uniquely determined by the image of s since the relation

$$a_1 = a_{1\cdot 1} = a_1 \circ {}^1a_1 = a_1 \circ a_1,$$

immediately implies that a_1 must be the identity. Moreover, the relation

$$\mathrm{id}_{X_{\mathbb{C}}} = a_1 = a_{s^2} = a_s \circ {}^s a_s,$$

tells us that $a = a_s$ must satisfy the relation ${}^s a = a^{-1}$. Thus, in this context, it is easy to explicitly compute the set $Z^1(\Gamma, A)$, and its quotient $H^1(\Gamma, A)$, provided one has a nice control on the group of automorphisms of $X_{\mathbb{C}}$. We will make use of this in the next section.

A result that will be crucial in the classification results that follow, is the following theorem, known as Hilbert's Theorem 90.

Theorem 3.9. *Let L/K be a finite Galois extension with Galois group Γ. Consider the multiplicative group L^* with its natural Γ-action. Then $H^1(\Gamma, L^*) = 0$.*

This one we can actually prove.

Proof. Let $a \in Z^1(\Gamma, L^*)$ be a cocycle. We need to prove that $a \sim 1$, that is, there exists an element $b \in L^*$ such that $a_s = b\,{}^s b^{-1}$ for every $s \in \Gamma$.

Considering $s \in \Gamma$ as a function $L \to L$, we may consider the (K-linear) function

$$f := \sum_{s \in \Gamma} a_s \cdot s.$$

By linear independence of automorphisms, we know that this function is nontrivial, which means that there exists $c \in L$ such that $f(c) \neq 0$. Define $b = f(c) = \sum_{s \in \Gamma} a_s {}^s c$ and note that

$${}^s b = \sum_{t \in \Gamma} {}^s a_t {}^{st} c = \sum_{t \in \Gamma} a_s^{-1} a_{st} {}^{st} c = a_s^{-1} b.$$

This means that $a_s = b\,{}^s b^{-1}$, as wished. ☺

4. Torsors (Lecture 2)

We keep the notations from last section: L/K is a finite Galois extension with Galois group Γ. We already saw that the Galois cohomology set $H^1(\Gamma, A)$ is extremely related to K-forms of an L-variety with automorphism group A, and essentially any algebraic object with that automorphism group (over L).

One algebraic object that fulfills this vague definition and is thus intimately related with cohomology is the notion of a torsor.

Definition 4.1. Let G be an algebraic K-group. A (right) G-torsor is an algebraic K-variety X, equipped with an algebraic (right) G-action $a : X \times G \to X$ such that there exists a finite Galois extension L/K for which X_L is isomorphic to G_L and the action becomes multiplication on the right. That is, there exists an isomorphism of L-varieties $f : X_L \to G_L$ and a commutative diagram

$$\begin{array}{ccc} X_L \times G_L & \xrightarrow{a_L} & X_L \\ \Big\downarrow{\scriptstyle \mathrm{id}_{G_L} \times f} & & \Big\downarrow{\scriptstyle f} \\ G_L \times G_L & \xrightarrow{m} & G_L, \end{array}$$

where a_L denotes the action a seen over L (i.e. same equations, bigger field) and m denotes mutiplication in G_L. We say that L trivializes the torsor.

A morphism of torsors is a G-equivariant morphism $X \to Y$. One can prove that such a morphism is always an isomorphism.

The obvious example of a torsor is the trivial one, which corresponds to G acting on $X = G$ by right multiplication. In this case, one can take $L = K$.

Remark 4.2. Note that a G-torsor is by definition a K-form of the *variety* G_L. But unless it is the trivial torsor, it will not be an algebraic K-group.

Actually, nontrivial torsors are hard to fathom in a naive way, since they do not have K-points at all! Indeed, if a G-torsor X has a K-point p, one can then define an isomorphism $G \to X$ via $g \mapsto p \cdot g$, which is clearly G-equivariant and thus proves that the torsor is trivial.

Torsors are thus a perfect example of the fact that one cannot understand a K-variety simply by looking at its K-points. The defining equations matter and their solutions over fields L/K matter too. That being said, let us give a concrete example of a nontrivial torsor.

Example 4.3. Consider the real torus $\mathbb{S}^1$ given by the equation $x^2+y^2=1$. Recall that its multiplication is given by the formula for complex multiplication

$$(x, y) \cdot (z, w) = (xz - yw, xw + yz).$$

Consider then the "pointless" conic C, defined by $x^2 + y^2 = -1$. Then the same formula above defines an action $\mathbb{S}^1 \times C \to C$ that turns C into an $\mathbb{S}^1$-torsor. Note that over $\mathbb{C}$ both varieties are isomorphic, since we can simply make a change of variables $x' = ix$ and $y' = iy$. This is of course not possible over $\mathbb{R}$.

Replacing -1 in the formula above by any other negative real number yields another $\mathbb{S}^1$-torsor, but it is isomorphic to C. Indeed, an isomorphism between $x^2 + y^2 = -a$ with $a > 0$ and C can be given by simply applying the change of variables $x' = \sqrt{a}x$ and $y' = \sqrt{a}y$, which is an isomorphism of $\mathbb{R}$-varieties since $\sqrt{a} \in \mathbb{R}$. One can check directly that this isomorphism is $\mathbb{S}^1$-equivariant.

Let us give now the explicit relation between torsors and cohomology.

Proposition 4.4. *Let K be a field and G be an algebraic K-group. Let L/K be a finite Galois extension. Then G-torsors trivialized by L/K are classified by $H^1(\Gamma, G(L))$.*

We cannot prove this result here since, once again, it needs tools from Galois descent. But we can at least give an explicit description of the map relating the two notions.

Let X be a G-torsor trivialized by L. Then it admits an L-point p. Since the action over L is transitive, for every $s \in \Gamma$ there exists $a_s \in G(L)$ such that ${}^s p = p \cdot a_s$. Then, for $s, t \in \Gamma$ we have

$$p \cdot a_{st} = {}^{st}p = {}^s({}^t p) = {}^s(p \cdot a_t) = {}^s p \cdot {}^s a_t = (p \cdot a_s) \cdot {}^s a_t = p \cdot (a_s {}^s a_t),$$

from where we get that $a_{st} = a_s {}^s a_t$ since the action has trivial stabilizers. One can check that the choice of another point gives a cocycle that is equivalent to this one, so that we get a well-defined map

$$\{G\text{-torsors trivialized by } L\} \to H^1(\Gamma, G(L)).$$

Another way of understanding this result is by noting that any G-torsor is a K-form of the trivial G-torsor *as a G-variety*. Then, by the general principle lying behind Theorem 3.6, those trivialized by L must be classified by $H^1(\Gamma, A)$, where A denotes the group of automorphisms *of G-varieties* over L of the trivial G-torsor G_L. It is not hard to see that the only such automorphisms correspond to multiplication by an L-point of G on the left (recall that the action on a G-torsor is on the right), so that A is actually $G(L)$.

Example 4.5. The K-torus $\mathbb{G}_m$ has no nontrivial torsors. Indeed, by Hilbert's Theorem 90, we know that $H^1(\Gamma, L^*)$ is always trivial, for any finite extension L/K.

Example 4.6. The real torus $\mathbb{S}^1$ has only one nontrivial torsor. Indeed, recall that $\mathbb{S}^1$ is obtained via the action of $\Gamma = \{1, s\}$ on $\mathbb{C}^*$ given by ${}^s z = \bar{z}^{-1}$, so that its fixed points are indeed those of the real circle. Thus, we need to compute $H^1(\Gamma, \mathbb{C}^*)$ for this particular action.

Let us compute $Z^1(\Gamma, \mathbb{C}^*)$ first. As we mentioned in Example 3.8, this amounts to finding those $z \in \mathbb{C}^*$ such that ${}^s z = z^{-1}$. Given the action of Γ, this can only occur if $z \in \mathbb{R}^*$. Now, for any $a \in \mathbb{R}^*$, we see that ${}^s a = a^{-1}$, so that $a^s a^{-1} = a^2$. Hence a cocycle given by $z \in \mathbb{R}^*$ will be equivalent to one given by $az^s a^{-1} = a^s a^{-1} z = a^2 z$, so that there are at most two equivalence classes: that of $z = 1$ and that of $z = -1$. This tells us that there are at most two nonisomorphic torsors and hence at most one that is nontrivial. Since we already know such an example, we are done.

5. Classification of real tori and their torsors (Lecture 3)

Definition 5.1. Let K be a field. A K-torus is an algebraic K-group that is isomorphic to the split torus $\mathbb{G}_m^n$ over a finite extension L/K (which may be assumed to be Galois). In other words, a K-torus is a K-form of the split torus.

By Theorem 3.6, tori split by L/K are classified by the group $H^1(\Gamma, A)$, where Γ is the Galois group of L/K and $A = \mathrm{Aut}_{L-\mathrm{gp}}(\mathbb{G}_m^n) \simeq \mathrm{GL}_n(\mathbb{Z})$. Since this last isomorphism comes from considering the exponents on the variables involved, one can prove that it commutes with the action of Γ on $\mathbb{G}_m^n = (L^*)^n$. This implies that Γ acts trivially on A, and hence computing $H^1(\Gamma, A)$ becomes in fact quite easier.

First, we see that the cocycle condition becomes $a_{st} = a_s \circ a_t$, which implies that we are looking at homomorphisms $\Gamma \to A$. Moreover, the equivalence relation becomes $b_s = \varphi \circ a_s \circ \varphi^{-1}$ for $\varphi \in A$. In other words, it is the equivalence relation given by conjugation. Computing $H^1(\Gamma, A)$ amounts then to computing morphisms to $\mathrm{GL}_n(\mathbb{Z})$ up to conjugation. Equivalently, one can classify Γ-actions on $\mathbb{Z}^n$ up to base change. This equivalence can be made explicit when looking at the module of characters $M = \mathrm{Hom}(T_L, \mathbb{G}_m)$ of the torus T_L, which is isomorphic to $\mathbb{Z}^n$ as a group, or its dual $N = \mathrm{Hom}(\mathbb{G}_{m,\mathbb{C}}, T_{\mathbb{C}}) = \mathrm{Hom}(M, \mathbb{Z})$, the module of cocharacters.

In the particular case of $K = \mathbb{R}$, the only nontrivial choice for L is $\mathbb{C}$. Then, by the previous discussion, we see that classifying real tori amounts to classifying $\mathbb{Z}/2\mathbb{Z}$-actions on $\mathbb{Z}^n$. This yields the following result.

Theorem 5.2. *Up to isomorphism, every real torus is a direct product of copies of the following three tori: $\mathbb{G}_{m,\mathbb{R}}$, $\mathbb{S}^1$ and $R_{\mathbb{C}/\mathbb{R}}(\mathbb{G}_{m,\mathbb{C}})$. Their corresponding groups of cocharacters N are $\mathbb{Z}$ with trivial action of Γ, $\mathbb{Z}$ with action of Γ by ± 1, and $\mathbb{Z}^2$ with action of Γ by permutation of the coordinates.*

Let us state more precisely the result in terms of modules. The result above follows immediately from this.

Proposition 5.3. *Let $\Gamma = \{1, s\}$ act on $M = \mathbb{Z}^n$. Then M decomposes as a direct sum of copies of the following three submodules: $M_0 = \mathbb{Z}$ with trivial action, $M_1 = \mathbb{Z}$ with action by ± 1, and $M_2 = \mathbb{Z}^2$ with action by permutation of coordinates.*

The proof of this result is elementary, but a bit complicated, so we will only treat the case $n = 2$.

Proof of the case $n = 2$. Consider the element s as an element of order two in $\mathrm{GL}_2(\mathbb{Z})$. Then its eigenvalues fall into one of the following three cases:

- The only eigenvalue is 1, in which case the action is trivial and $M \simeq M_0 \oplus M_0$.
- The only eigenvalue is -1, in which case we have $M \simeq M_1 \oplus M_1$.
- The eigenvalues are 1 and -1, in which case the characteristic polynomial of s is $x^2 - 1$. We see then that the matrix must have trace 0 and determinant -1. In other words,
$$s = \begin{pmatrix} a & b \\ c & -a \end{pmatrix}, \quad \text{with } a, b, c \in \mathbb{Z} \text{ and } a^2 + bc = 1.$$
We have then two subcases:
  - If $a = 0$, then $bc = 1$, which means that either e_1 and e_2 are being exchanged, or e_1 and $-e_2$ are being exchanged. In both cases, $M \simeq M_2$.
  - If $a = 1$, then $bc = 0$. Then $b = c = 0$ since otherwise s would not have order 2. This implies that $M \simeq M_0 \oplus M_1$.

☺

We already saw in Examples 4.5 and 4.6 how to classify torsors for $\mathbb{G}_{\mathrm{m},\mathbb{R}}$ and $\mathbb{S}^1$. And from the cocycle definition of H^1 we immediately see that
$$H^1(\mathbb{R}, T_1 \times T_2) = H^1(\mathbb{R}, T_1) \times H^1(\mathbb{R}, T_2),$$
so that in order to get a full classification of torsors, it only remains to classify torsors for $R_{\mathbb{C}/\mathbb{R}}(\mathbb{G}_{\mathrm{m},\mathbb{C}})$.

Recall that $R_{\mathbb{C}/\mathbb{R}}(\mathbb{G}_{\mathrm{m},\mathbb{C}})$ is obtained via the action of $\Gamma = \{1, s\}$ on $(\mathbb{C}^*)^2$ given by ${}^s(z_1, z_2) = (\bar{z}_2, \bar{z}_1)$. Let us compute $H^1(\Gamma, (\mathbb{C}^*)^2)$. As we have seen before, a cocycle amounts to a pair $(z_1, z_2) \in (\mathbb{C}^*)^2$ such that ${}^s(z_1, z_2) = (z_1, z_2)^{-1}$. Given the action of Γ, this occurs if and only if $z_2 = \bar{z}_1^{-1}$, in which case
$$(z_1, z_2) = (z_1, \bar{z}_1^{-1}) = (z_1, 1)(1, \bar{z}_1^{-1}) = (z_1, 1)^s (z_1, 1)^{-1},$$
so that every cocycle is equivalent to $\mathbf{1}$ and hence $H^1(\Gamma, R_{\mathbb{C}/\mathbb{R}}(\mathbb{G}_{\mathrm{m},\mathbb{C}})) = 1$.

All in all, we have proved the following result.

Proposition 5.4. *If T is a direct product of l copies of $\mathbb{G}_{\mathrm{m},\mathbb{R}}$, m copies of $\mathbb{S}^1$ and n copies of $R_{\mathbb{C}/\mathbb{R}}(\mathbb{G}_{\mathrm{m},\mathbb{C}})$, then $H^1(\Gamma, T) \simeq (\mathbb{Z}/2\mathbb{Z})^m$.*

6. Real forms of toric varieties (Lecture 3)

Let K be a field (of characteristic 0). The notion of toric variety can be easily extended to this context.

Definition 6.1. A toric variety is a normal T-variety X, for some K-torus T, that admits a dense open orbit.

An immediate consequence of this definition is that, if we pass to the algebraic closure $\bar{K}$ of K, then we recover the original definition of a toric variety. In practice, it suffices to go a finite extension that splits the torus T in order to compute everything from the combinatorial data defining X (note that all the constructions from Section 1 work over *any* field). The machinery of K-forms is then well-suited to study toric varieties over an arbitrary field.

Definition 6.2. Let T be a K-torus split by L/K and let X be a toric T_L-variety. A K-form of X is a T-variety Y such that $X \simeq Y_L$ as T_L-varieties. In particular, Y is a toric T-variety.

For all practical purposes, one may assume that the action of the torus is faithful, i.e. there is no nontrivial subgroup of T acting trivially on X (otherwise one could replace T by its quotient by the corresponding subgroup). Then another consequence of the definition is that, if we reduce the action of T to the dense open orbit, we recover a T-torsor. In particular, T-torsors are examples of toric varieties.

Let T be a K-torus, let L/K be a Galois extension with Galois group Γ splitting T. Recall that Γ acts naturally on the group of cocharacters $N = \mathrm{Hom}(\mathbb{G}_{\mathrm{m},\mathbb{C}}, T_{\mathbb{C}})$ and the group of characters $M = \mathrm{Hom}(T_{\mathbb{C}}, \mathbb{G}_{\mathrm{m},\mathbb{C}}) = \mathrm{Hom}(N, \mathbb{Z})$. In particular, it will naturally move cones on any fan defined over $N_{\mathbb{R}}$.

The following result, put forth by Matthieu Huruguen in his PhD thesis, gives a complete answer to the question on existence of real forms of toric varieties.

Theorem 6.3. *Let T be a real torus, let $\Gamma = \{1, s\}$ be the Galois group of $\mathbb{C}/\mathbb{R}$, let $N = \mathrm{Hom}(\mathbb{G}_{\mathrm{m},\mathbb{C}}, T_{\mathbb{C}})$ be the group of characters of $T_{\mathbb{C}}$ and let Σ be a fan in $N_{\mathbb{R}}$, defining a complex toric $T_{\mathbb{C}}$-variety X. Then X admits a real form if and only if the fan Σ is Γ-stable, i.e. for every $\sigma \in \Sigma$. $s(\sigma)$ is also a cone in Σ.*

Moreover, when this hypothesis is satisfied, the map

$$\{\textit{real forms of } X\} \to \{T\textit{-torsors}\},$$

that sends a real form to its dense open orbit, is bijective.

Remark 6.4. The second statement generalizes to any Galois extension L/K splitting the torus T. The first one needs one extra technical hypothesis on the action of Γ on the fan, which is trivially satisfied when Γ has order 2.

Thus, in order to classify real forms of toric varieties, we must follow three steps:

(i) Classify real tori (i.e. real forms of $\mathbb{G}^n_{\mathrm{m},\mathbb{C}}$).
(ii) For each real torus T, classify its torsors.
(iii) For each real torus T, classify the Γ-stable fans for the induced action of Γ on $N_{\mathbb{R}}$.

Let us carry on this program in small dimensions.

6.1. **Real forms of 1-dimensional toric varieties.** Over $\mathbb{C}$, there are three different 1-dimensional fans and hence three toric varieties of dimension 1: the torus $\mathbb{G}_{\mathrm{m},\mathbb{C}} = \mathbb{C}^*$ itself, the affine line $\mathbb{A}^1_{\mathbb{C}} = \mathbb{C}$, and the projective line $\mathbb{P}^1_{\mathbb{C}}$. However, there are two one-dimensional real tori: $\mathbb{G}_{\mathrm{m},\mathbb{R}}$ and $\mathbb{S}^1$. In order to get a full classification we must see, for each one of the three toric varieties, whether the fan is Γ-stable for the two different Γ-actions on N (given by the two different tori), and then we must take torsors into account.

Let us start with the real torus $\mathbb{G}_{\mathrm{m},\mathbb{R}}$. Then the Γ-action on N is trivial, so that any fan and any $\mathbb{G}_{\mathrm{m},\mathbb{R}}$-torsor will yield a real toric variety. Now, all $\mathbb{G}_{\mathrm{m},\mathbb{R}}$-torsors are trivial by Example 4.5, so that we are only left with the fans. We obtain then three real toric varieties: $\mathbb{G}_{\mathrm{m},\mathbb{R}} = \mathbb{R}^*$, $\mathbb{A}^1_{\mathbb{R}} = \mathbb{R}$ and $\mathbb{P}^1_{\mathbb{R}}$.

Moving on to the torus $\mathbb{S}^1$, we see that the Γ-action on N is by ± 1, so that the fans of $\mathbb{G}_{\mathrm{m},\mathbb{C}}$ and $\mathbb{P}^1_{\mathbb{C}}$ are Γ-stable, but the fan of $\mathbb{A}^1_{\mathbb{C}}$ is not. Moroever, recall that $\mathbb{S}^1$

has exactly two torsors: the trivial one ($\mathbb{S}^1$) and the nontrivial one (the "pointless" conic C). These correspond to the two forms of $\mathbb{G}_{\mathrm{m},\mathbb{C}}$ as an $\mathbb{S}^1$-variety.

But we should also get two forms of $\mathbb{P}^1_{\mathbb{C}}$ as $\mathbb{S}^1$-varieties, and we know that $\mathbb{P}^1_{\mathbb{C}}$ admits two real forms as a variety already. One of the two forms must contain the "pointless" conic C as an open dense subset. It cannot be $\mathbb{P}^1_{\mathbb{R}}$, since it has a lot of real points, so it must be the projective "pointless" conic $\bar{C}$. Conversely, the trivial torsor $\mathbb{S}^1$, which has real points, must be a dense open subset of the other real form, so this one must be $\mathbb{P}^1_{\mathbb{R}}$. We learn then that $\mathbb{S}^1$ can be naturally embedded into $\mathbb{P}^1_{\mathbb{R}}$, compatibly with the action on itself by left multiplication. With some work, one can discover that this embedding is actually given by the stereographic projection

$$\mathbb{S}^1 \hookrightarrow \mathbb{P}^1_{\mathbb{R}} : (x,y) \mapsto \begin{cases} [x : 1-y] & \text{if } (x,y) \neq (0,1); \\ [1:0] & \text{if } (x,y) = (0,1); \end{cases}$$

where (x, y) is a solution of $x^2 + y^2 = 1$. Note that this defines a bijection $\mathbb{S}^1(\mathbb{R}) \to \mathbb{P}^1_{\mathbb{R}}(\mathbb{R})$ on *real* points, but if we extend this formula to $\mathbb{C}$-points, then we miss exactly two (complex) points on the image: $[i : 1], [-i : 1] \in \mathbb{P}^1_{\mathbb{C}}$ (Exercise!). This makes sense since, over $\mathbb{C}$, we are simply embedding $\mathbb{G}_{\mathrm{m},\mathbb{C}} = \mathbb{C} \smallsetminus \{0\}$ in $\mathbb{P}^1_{\mathbb{C}} = \mathbb{C} \cup \{\infty\}$, so we must miss exactly two points.

In summary, real toric varieties of dimension one, up to isomorphism, are the following:

- the $\mathbb{G}_{\mathrm{m},\mathbb{R}}$-varieties $\mathbb{G}_{\mathrm{m},\mathbb{R}}$, $\mathbb{A}^1_{\mathbb{R}}$ and $\mathbb{P}^1_{\mathbb{R}}$;
- the $\mathbb{S}^1$-varieties $\mathbb{S}^1$, C, $\mathbb{P}^1_{\mathbb{R}}$ and $\bar{C}$.

Remark 6.5. Note that $\mathbb{P}^1_{\mathbb{R}}$ appears twice. This means that the same algebraic variety can be acted upon by different tori. Then, as toric varieties, they are different.

6.2. **Real forms of toric surfaces.** Here things get wilder since, already over $\mathbb{C}$, there are infinitely many different 2-dimensional fans. However, there are still only a few 2-dimensional real tori: $\mathbb{G}^2_{\mathrm{m},\mathbb{R}}$, $(\mathbb{S}^1)^2$, $\mathbb{G}_{\mathrm{m},\mathbb{R}} \times \mathbb{S}^1$ and $R_{\mathbb{C}/\mathbb{R}}(\mathbb{G}_{\mathrm{m},\mathbb{C}})$. Moreover, both $\mathbb{G}^2_{\mathrm{m},\mathbb{R}}$ and $R_{\mathbb{C}/\mathbb{R}}(\mathbb{G}_{\mathrm{m},\mathbb{C}})$ do not have nontrivial torsors, while $\mathbb{G}_{\mathrm{m},\mathbb{R}} \times \mathbb{S}^1$ has only one nontrivial torsor (which is $\mathbb{G}_{\mathrm{m},\mathbb{R}} \times C$) and $(\mathbb{S}^1)^2$ has three (C^2, $C \times \mathbb{G}_{\mathrm{m},\mathbb{R}}$ and $\mathbb{G}_{\mathrm{m},\mathbb{R}} \times C$).

Example 6.6 ($\mathbb{G}^2_{\mathrm{m},\mathbb{R}}$-varieties)**.** Any fan on $N_{\mathbb{R}} \simeq \mathbb{R}^2$ gives rise to a unique $\mathbb{G}^2_{\mathrm{m},\mathbb{R}}$-variety. Indeed, in this case the Γ-action is trivial and hence every fan is Γ-stable, and there are no nontrivial torsors here. In particular we may recover the real varieties $\mathbb{P}^2_{\mathbb{R}}$ and $\mathrm{Bl}_p(\mathbb{P}^2_{\mathbb{R}}) = \mathbb{F}_1$. Indeed, blow-ups are well-defined over an arbitrary field K, as long as one blows-up a K-point (which is the case here: we blew up the real point $p = [1 : 0 : 0]$).

This reminds us that the classical theory of toric varieties does indeed extend to an arbitrary field K with no problems, as long as we consider the torus $\mathbb{G}^n_{\mathrm{m}}$ and not a K-form of it.

In order to continue, it is useful to note that:

- For $\mathbb{G}^2_{\mathrm{m},\mathbb{R}}$, the action of Γ on $N_{\mathbb{R}} \simeq \mathbb{R}^2$ is given by the trivial action.
- For $(\mathbb{S}^1)^2$, the action of Γ on $N_{\mathbb{R}} \simeq \mathbb{R}^2$ is given by multiplication by ± 1.
- For $\mathbb{G}_{\mathrm{m},\mathbb{R}} \times \mathbb{S}^1$, the action of Γ on $N_{\mathbb{R}} \simeq \mathbb{R}^2$ is given by $e_1 \mapsto e_1$ and $e_2 \mapsto -e_2$.
- For $R_{\mathbb{C}/\mathbb{R}}(\mathbb{G}_{\mathrm{m},\mathbb{C}})$, the action of Γ on $N_{\mathbb{R}} \simeq \mathbb{R}^2$ is given by $e_1 \mapsto e_2$ and $e_2 \mapsto e_1$.

Example 6.7 (Forms of the projective plane $\mathbb{P}^2_{\mathbb{C}}$)**.** Let us focus the toric variety $\mathbb{P}^2_{\mathbb{C}}$, whose fan is given by the rays

$$\langle e_1\rangle,\ \langle e_2\rangle,\ \text{and}\ \langle -e_1-e_2\rangle,$$

and the corresponding 2-dimensional cones. We see that this fan has no central symmetry and is not symmetric with respect to the x-axis, but it is symmetric with respect to the diagonal axis. This tells us that this variety cannot descend to an $(\mathbb{S}^1)^2$-variety or a $(\mathbb{G}_{m,\mathbb{R}}\times\mathbb{S}^1)$-variety, but it does admit a unique structure of $R_{\mathbb{C}/\mathbb{R}}(\mathbb{G}_{m,\mathbb{C}})$-variety (uniqueness comes from the fact that this torus has no nontrivial torsors).

With a suitable change of variables, one can prove that this is once again the variety $\mathbb{P}^2_{\mathbb{R}}$. We see then that $\mathbb{P}^2_{\mathbb{C}}$ is another example of a variety that admits actions of several tori, but not many real forms (they are always $\mathbb{P}^2_{\mathbb{R}}$).

Example 6.8 (Forms of $\mathbb{F}_1$)**.** Consider now the toric variety $\mathbb{F}_1$, whose fan is given by the rays

$$\langle e_1\rangle,\ \langle e_2\rangle,\ \langle -e_1\rangle\ \text{and}\ \langle -e_1-e_2\rangle,$$

while the 2-dimensional cones are

$$\langle e_1,e_2\rangle,\ \langle e_2,-e_1\rangle,\ \langle -e_1,-e_1-e_2\rangle\ \text{and}\ \langle -e_1-e_2,e_1\rangle.$$

Here things get actually quite simpler. A quick look at the drawing of this fan tells us that there are no symmetries whatsoever, which implies that this variety does not admit any structure of $(\mathbb{S}^1)^2$-variety, $(\mathbb{G}_{m,\mathbb{R}}\times\mathbb{S}^1)$-variety or $R_{\mathbb{C}/\mathbb{R}}(\mathbb{G}_{m,\mathbb{C}})$-variety. Thus, we must only care about the unique form as a $\mathbb{G}^2_{m,\mathbb{R}}$-variety, which is $\mathbb{F}_1$.

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