On singular del Pezzo surfaces embedded in weighted projective spaces

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Section 1

Weighted del Pezzo surfaces

Objects

Definition 1

$$S_d \subset \mathbb{P}(a_0, a_1, a_2, a_3),$$

that is, hypersurfaces in weighted projective space given by

$$f(x, y, z, w) = 0 \subset \mathbb{P}(a_0, a_1, a_2, a_3),$$

where f is quasi- homogeneous polynomial.

1. S_d is well-formed
   elements of the set $\{a_0, a_1, a_2, a_3\} \setminus a_i$ is mutually prime for any i.
2. S_d is quasi-smooth, which means $f(x, y, z, w) = 0$ has singularities only at the origin in $\mathbb{C}^4$.

Main problem

- K-stabilty of the surface (K-E surfaces).
- The existence of the $-K$-polar cylinder on the surfaces.

Motivation of K-stability of the surfaces

The existence problem of Kähler-Einstein metric on weighted del Pezzo surfaces itself.

Sasaki-Einstein metric on 5-manifolds

$\updownarrow$

Kähler-Einstein metric on weighted del Pezzo surfaces

Sasakian manifolds

A Riemannian manifold (M, g) is called Sasakian if its conical metric $\bar{g} = r^2 g + dr^2$ is a Kähler metric on the cone $C(M) = M \times \mathbb{R}^+$.

An odd dimensional counterpart of Kähler metrics, which are defined on even dimensional manifolds.

(Smale, Barden)(1962) Closed simply connected 5-manifold are completely classified.

Every closed simply connected Sasaki-Einstein 5-manifold is spin.

Classification

Theorem 1.1

(Smale 1962) For a positive integer m, there is a unique closed simply connected 5-dimensional manifold M_m with $\mathrm{H}_2(M_m, \mathbb{Z}) = \mathbb{Z}/m\mathbb{Z} \oplus \mathbb{Z}/m\mathbb{Z}$ that admits a spin structure. Furthermore, a closed simply connected 5-dimensional manifold M that admits a spin structure is of the form

$$M = k(S^2 \times S^3) \# M_{m_1} \# \ldots \# M_{m_r},$$

where $k(S^2 \times S^3)$ is the k-fold connected sum of $S^2 \times S^3$ for a non-negative integer k and m_i is a positive integer greater than 1 with m_i dividing m_{i+1}.

Classification

3-types

- Rational homology sphere,
  $k = 0$,that is, $M = M_{m_1} \# \ldots \# M_{m_r}$
- Torsion free with positive second betti number,
  $M = k(S^2 \times S^3) = kM_\infty$
- Mixed type,
  $k \neq 0$ and torsion part survives.

Torsion free

Theorem 1.2

(Boyer, Galicki, Nakamaye, Kollár) $M = k(S^2 \times S^3) = kM_\infty$ *has Sasaki-Einstein metric*

Linked with successive blow ups of $\mathbb{P}^1 \times \mathbb{P}^1$

Open problem since 2005
The existence of Sasaki-Einstein metric on nM_2?

Theorem 1.3

(Park, W- 2021)
For each positive integer $n \geq 4$, the rational homology 5-sphere nM_2 admits a Sasaki-Einstein metric.

We have complete classification of Sasaki-Einstein metric on Rational homology 5-spheres

$$\begin{array}{ccc} L_{S_d} & \hookrightarrow & S^{2n+1} \\ \downarrow & & \downarrow \\ S_d & \hookrightarrow & \mathbb{P} \end{array}$$

L_{S_d} is Sasaki-Einstein
S_d is Kähler-Einstein

S_{4n+2} : hypersurface of degree $4n+2$ in $\mathbb{P}(2,2,2n,2n+1) = \mathbb{P}(x,y,z,w)$

$$L_{S_{4n+2}} = nM_2$$

S_{4n+2} is not well-formed!

S_{2n+1} : hypersurface of degree $2n+1$ in $\mathbb{P}(1,1,n,2n+1)$

Lemma 1.4

If there is a Kähler-Einstein edge metric on S_{2n+1} with angle $\frac{\pi}{m}$ along the divisor C_w, then there is a Sasaki-Einstein metric on the link nM_2 of S_{4n+2}.

Sasaki-Einstein metric on nM_2

Theorem 1.5

(Blum, Jonsson, Odaka, Fujita 2020) Fano pair (X, D) is K-stable if and only if $\delta(X, D) > 1$.

$$\delta(S_{2n+1}, \frac{1}{2} C_w) \geq \frac{8n+8}{8n+7}.$$

Sasaki-Einstein metric on nM_2

$(S_{2n+1}, \frac{1}{2}C_w)$ is log K-stable

$\longrightarrow S_{2n+1}$ has Kähler-Einstein edge metric with angle $\frac{\pi}{m}$ along the divisor C_w
by (Li,Tian, Wang 2019)

$\longrightarrow$ There is a Sasaki-Einstein metric on the link nM_2 of S_{4n+2}.

S_{4n+2} : hypersurface of degree $4n+2$ in $\mathbb{P}(2,2,2n,2n+1) = \mathbb{P}(x,y,z,w)$

S_{2n+1} : hypersurface of degree $2n+1$ in $\mathbb{P}(1,1,n,2n+1)$
$(S_{2n+1}, C_w) = (\mathbb{P}(1,1,n), C)$, where C is a curve of degree $2n+1$
The link of $S_{4n+2} \sim (S_{2n+1}, C_w) = (\mathbb{P}(1,1,n), C)$= nM_2

Theorem 1.6

(Kollar 2005) The link of $(\mathbb{F}_n, \frac{1}{2}C + \frac{1}{m}e)$*, where* $C \in |\, 2e + (2n+1)f \,|$ *is* $M_\infty \# nM_2$*. And it is Sasaki-Einstein.*

Theorem 1.7

(Jeong, Kim, Park, W- 2022) The link of a degree $4(2n+1)$ *hypersurface of* $\mathbb{P}(2,4,4n,4n+1)$ *is* $2M_\infty \# nM_2$*. And it is Sasaki-Einstein.*

A degree $4(2n+1)$ hypersurface of $\mathbb{P}(2,4,4n,4n+1)$ $\sim (S_{4n+2}, 1/2\, W)$, where $S_{4n+2} \subset \mathbb{P}(1,2,2n,4n+1)$ and $W = \{w=0\}$ is a curve of degree $4n+2$ in $\mathbb{P}(1,2,2n)$.

$$k(>2)M_\infty \# nM_2 ?????????$$

Section 2

K-stability of Weighted del Pezzo surfaces

K-E on del Pezzo hypersurfaces

$S_d \subset \mathbb{P}(a_0, a_1, a_2, a_3)$
Fano index $I = a_0 + a_1 + a_2 + a_3 - d$

Johnson and Kollár found all possibilities of (a_0, a_1, a_2, a_3, d) for $I = 1$.

- $(a_0, a_1, a_2, a_3, d) = (2, 2m+1, 2m+1, 4m+1, 8m+4)$ for some integer m
- $(1,1,1,1,3), (1,1,1,2,4), (1,1,2,3,6), (1,2,3,5,10), (1,3,5,7,15), (1,3,5,8,16), (2,3,5,9,18), (3,3,5,5,15), (3,5,7,11,25), (3,5,7,14,28), (3,5,11,18,36), (5,14,17,21,56), (5,19,27,31,81), (5,19,27,50,100), (7,11,27,37,81), (7,11,27,44,88), (9,15,17,20,60), (9,15,23,23,69), (11,29,39,49,127), (11,49,69,128,256), (13,23,35,57,127), (13,35,81,128,256)$

And Johnson and Kollár proved the existence of the orbifold Kähler-Einstein metrics except the following four quintuples (2001)

$$(1, 2, 3, 5, 10), \quad (1, 3, 5, 7, 15), \quad (1, 3, 5, 8, 16), \quad (2, 3, 5, 9, 18).$$

Araujo(2002) computed the alpha invariants for the two of these four cases to show the existence of an orbifold Kähler-Einstein metric

$$(1, 2, 3, 5, 10),$$

$(1, 3, 5, 7, 15)$ and the defining equation contains the monomial yzt

K-E on del Pezzo hypersurfaces

For the case $I = 1$ Cheltsov, Shramov and Park (2010) proved that every log del Pezzo hypersurface admits an orbifold Kähler-Einstein metric except possibly the case when $(1, 3, 5, 7, 15)$ and the defi ning equation does not contain the monomial yzt

Cheltsov, Shramov and Park (2018) every log del Pezzo hypersurface with $I = 1$ admits an orbifold Kähler-Einstein.

Higher index cases

Theorem 2.1

(Paemurru 2016) For every $I \geq 1$, there is an algorithm which classifies all possible hypersurfaces of index I.

Obstruction to be Kähler-Einstein

Theorem 2.2

(Gauntlett, Martelli, Sparks, Yau 2007) If $I > 3a_0$ then the corresponding del Pezzo hypersurface is not Kähler-Einstein

Theorem 2.3

(Alper, Blum, Halpern-Leistner, Xu 2020) If (X, D) is a K-polystable log Fano pair, then $Aut(X, D)$ is reductive.

Higher index cases

Example 2.4

$\mathbb{P}(1,1,2)$ is not Kähler-Einstein.
Since $(I = 4) > (3 = 3a_0)$
and also $Aut(\mathbb{P}(1,1,2))$ is not reductive.

Conjecture 2

(Cheltsov, Shramov and Park (2018)) If $I = 2, 3$ then S_d admits an orbifold Kähler-Einstein metric.

Index 2

Theorem 2.5

(Kim, W- 2021) Suppose that S_d *is quasi-smooth and the quintuple* (a_0, a_1, a_2, a_3, d) *is one of the following quintuples:*

$$(1, 6, 9, 13, 27), (1, 9, 15, 22, 45),$$

$$(1, 3, 3n + 3, 3n + 4, 6n + 9), (1, 3, 3n + 4, 3n + 5, 6n + 11),$$

$$(1, 1, n + 1, m + 1, n + m + 2), \text{ with } n < m$$

Then S_d *does not have an orbifold Kähler-Einstein metric.*

$I = 2$

Conjecture 1

A singular del Pezzo surface S_d with I = 2 and the quintuple (a_0, a_1, a_2, a_3, d) is not one of the above cases. Then the surface have an orbifold Kähler-Einstein metric.

Theorem 2.6

(Petracci, Liu) Let $a \geq 2$ be an integer and let X be a degree $2a$ quasi-smooth hypersurface in $\mathbb{P}(1, 1, a, a)$. Then X is K-polystable and not K-stable. Moreover, X admits a KE metric.

Theorem 2.7

(Kim, Viswanathan, W- 2022) The conjecture is true.

Among 44 cases in $I = 2$, 30 cases are previously known to exist.

No.	weights	degree	KE
1	$(1, 1, n+1, m+1)$	$n+m+2$	no if $n \neq m$
1′	$(1, 1, n, n)$	$2n$	yes
2	$(1, 2, n+2, n+3)$	$2n+6$	yes
3	$(1, 3, 3n+3, 3n+4)$	$6n+9$	no
4	$(1, 3, 3n+4, 3n+5)$	$6n+11$	no

No.	weights	degree	KE
5	$(1, 3, 4, 6)$	12	yes
6	$(1, 4, 5, 7)$	15	yes
7	$(1, 4, 5, 8)$	16	yes
8	$(1, 4, 6, 9)$	18	yes
9	$(1, 5, 7, 11)$	22	yes
10	$(1, 6, 9, 13)$	27	no
11	$(1, 6, 10, 15)$	30	yes
12	(1, 7, 12, 18)	36	yes
13	$(1, 8, 13, 20)$	40	yes
14	$(1, 9, 15, 22)$	45	no

$I > 2$

Project :
Classify all Kähler-Einstein nice hypersurfaces with $I \geq 3$

Theorem 2.8

(Kim, Viswanathan, W- 2023) Suppose that S is quasismooth and has index $I = 3$. Then S admits Kähler-Einstein metrics if and only if $I < 3a_0/2$

$I > 2$

- *class 1 tuples* $(a_0, a_1, b_2 + xm, b_3 + ym, b_2 + b_3 + (x + y)m)$ *where*
  - a_0, a_1 *are positive integers such that* $I = a_0 + a_1$,
  - $m = \operatorname{lcm}(a_0, a_1)$,
  - b_2, b_3 *are positive integers,*
- *class 2 tuples* $(a_0, a_1, a_2, b_3 + xm, a_1 + b_3 + xm)$ *where*
  - a_0, a_1, a_2 *are positive integers such that* $I = a_0 + a_2$ *and* $I > a_0 + a_1$,
  - $m = \operatorname{lcm}(a_0, a_1, a_2)$,
  - b_3 *is a positive integer,*
- *class 3 tuples* $(a_0, a_1, a_2, b_3 + xm, a_0 + b_3 + xm)$ *where*
  - a_0, a_1, a_2 *are positive integers such that* $I = a_1 + a_2$ *and* $I > a_0 + a_2$,
  - $m = \operatorname{lcm}(a_0, a_1, a_2)$,
  - b_3 *is a positive integer,*

$I > 2$

- *class 4 tuples* $(a_0, a_1, b_2 + xm, \frac{a_1}{2} + b_2 + xm, a_1 + 2b_2 + 2xm)$ *where*
  - a_0, a_1 *are positive integers such that* $I = a_0 + \frac{a_1}{2}$,
  - $m = \operatorname{lcm}\left(a_0, \frac{a_1}{2}\right)$,
  - b_2 *is a positive integer,*
- *class 5 tuples* $(a_0, a_1, b_2 + xm, \frac{a_0}{2} + b_2 + xm, a_0 + 2b_2 + 2xm)$ *where*
  - a_0, a_1 *are positive integers such that* $I = \frac{a_0}{2} + a_1$ *and* $I > a_0 + \frac{a_1}{2}$,
  - $m = \operatorname{lcm}\left(\frac{a_0}{2}, a_1\right)$,
  - b_2 *is a positive integer,*
- *class 6 tuples* $(a_0, a_1, b_2 + xm, b_3 + xm, d + 2xm)$
  $= (I - k, I + k, a + xm, a + k + xm, 2a + I + k + 2xm)$, *where*
  - I *is the index,*
  - k *is a positive integer such that* $I - k$ *is positive,*
  - $m = \operatorname{lcm}(a_0, a_1, k) = \operatorname{lcm}(I - k, I + k, k)$,
  - a *is a positive integer.*

+ some exotic families that does not belong to the list!

$I > 2$

Theorem 2.9

(Kim, W- 2025) We complete the classification of K-stability of weighted hypersurfaces.

Back to Sasaki-Einstein problem

Remaining families of mixed types :

(1) $k(S^2 \times S^3)\#2M_3, \quad k \geq 2$
(2) $k(S^2 \times S^3)\#3M_3, \quad k \geq 2$
(3) $k(S^2 \times S^3)\#nM_2, \quad k \geq 3$, $n > 2$
(4) $k(S^2 \times S^3)\#M_m, \quad k \geq 9$, $2 \leq m \leq 11$

Section 3

Cylindricity and exotic families

Ample Polar Cylinder

A cylinder in a projective variety is a Zariski open subset that is isomorphic to $\mathbb{A}^1 \times Z$, for some affine variety Z.

Definition 3.1

Let H be an $\mathbb{Q}$-divisor on a projective variety X. An H-polar cylinder in X is a Zariski open subset U of X such that

- $U \cong \mathbb{A}^1 \times Z$, for some affine variety Z, that is U is a cylinder in X,
- there is an effective $\mathbb{Q}$-divisor D on X with $D \equiv H$ and $U = X \setminus \mathrm{Supp}(D)$.

Such an effective divisor D is called the boundary of cylinder.

$(-K_{\mathbb{P}^2})$-polar Cylinder

Example

- For three lines L_1, L_2, L_3 meeting altogether only at a single point, $\mathbb{P}^2 \setminus \mathrm{Supp}(L_1 + L_2 + L_3) \cong \mathbb{A}^1 \times \mathbb{A}^1_{**}$.
- For a conic Q and a line L intersecting tangentially at a point, $\mathbb{P}^2 \setminus \mathrm{Supp}(Q + L) \cong \mathbb{A}^1 \times \mathbb{A}^1_{*}$.
- For a cuspidal cubic curve C and its Zariski tangent line T at the singular point, $\mathbb{P}^2 \setminus \mathrm{Supp}((1-\varepsilon)C + 3\varepsilon T) \cong \mathbb{A}^1 \times \mathbb{A}^1_{*}$.

$(-K)$-polar Cylinders in Smooth del Pezzo Surfaces

Theorem 3.2 (T. Kishimoto, Y. Prokhorov, M. Zaidenberg '11,'14)

- S_d *contains a* $(-K_{S_d})$*-polar cylinder if* $4 \leq d \leq 9$*.*
- S_d *does not contain any* $(-K_{S_d})$*-polar cylinder if* $d = 1, 2$*.*

Theorem 3.3 (I. Cheltsov, J. Park, W- '16)

S_3 *does not contain any* $(-K_{S_3})$*-polar cylinder.*

The absence of cylinders for cubic surfaces is proved by using the linear system on the boundary.

Theorem 3.4

S_d *contains a* $(-K_{S_d})$*-polar cylinder if and only if* $d \geq 4$*.*

Fact 1 : Unipotent Group Action

Theorem 3.5 (T. Kishimoto, Y. Prokhorov, M. Zaidenberg '13)

For a normal projective variety X with an ample divisor H, the corresponding generalized cone

$$\operatorname{Spec}\left(\bigoplus_{m\geq 0} \mathrm{H}^0\left(X, \mathcal{O}_X(mH)\right)\right)$$

admits nontrivial unipotent group action if and only if X contains an H-polar cylinder.

Relation between cylindricity and K-stability

Conjecture 2 (I. Kim, T. Okada, W- 2018)

X is a Fano variety with at most klt singularities and Pic$(X) = 1$. If X is birationally rigid, then X is K-stable.

Birationally rigid $\rightarrow$ non-rational.
In dimenseion 3, Rationality $\rightarrow$ Existence of Cylinder.
$(X, -K_X)$ has mild singularity (lct$(X) \geq 1$) $\rightarrow$ Non-existence of Cylinder.

Question 1

Question [I. Cheltsov, J. Park, Y. Prokhorov, M. Zaidenberg '21]

Let X be a Fano variety with at most klt singularities. If X does not contain any $(-K_X)$-polar cylinder, then X is K-polystable?

This holds for smooth del Pezzo surfaces, but it does not hold in general.

Theorem 3.6 (I. Kim, J. Kim, W- '24)

There are infinitely many del Pezzo surfaces with klt singularities which are K-unstable and do not contain any $(-K)$-polar cylinder.

Examples in exotic families in Paemurru list.

Question 2

Question

Is there a del Pezzo surface in a weighted projective space that contains a $(-K)$-polar cylinder?

To answer the question, we focused on the family of index 1.

Theorem 3.7 (I. Kim, J. Kim, W- '24)

For 62 quasi-smooth well-formed complete intersection log del Pezzo surfaces of index 1, there exists unique member that contains a $(-K)$*-polar cylinder.*

Paemurru List

Quasi-smooth well-formed del Pezzo hypersurfaces can be classified into eight families in terms of their indices $a_0 + a_1 + a_2 + a_3 - d$.

- Index 1 : 23 members
  $+39$ complete intersections $\Rightarrow$ totally 62 cases!
- Index 2 : 44 members
- Index 3 : 17 members
- Index 4 : 44 members
- Index 5 : 29 members
- Index 6 : 75 members

  $\vdots$
- 35 Infinite Series + 66 Sporadic Cases : Exotic families

Paemurru List

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- 35 Infinite Series + 66 Sporadic Cases : Exotic families
  $\Rightarrow$ Absence!

Paemurru List

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- Index 6 : 75 members

$\vdots$

- 35 Infinite Series + 66 Sporadic Cases : Exotic families

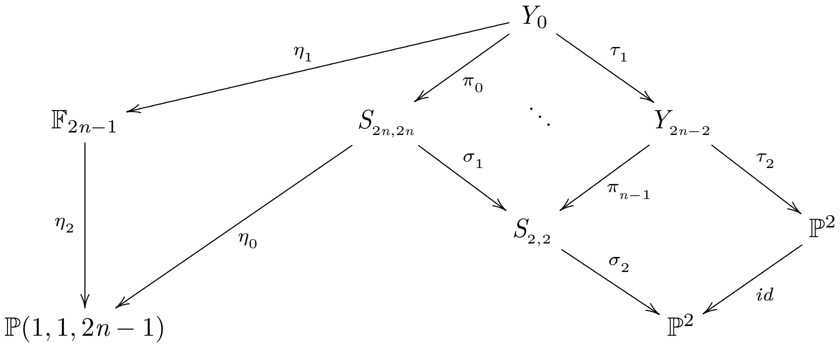
Main Result

Theorem 3.8 (I. Kim, J. Kim, W– '24)

Let S be one of the 62 quasi-smooth well-formed complete intersection log del Pezzo surfaces of index 1.

- $\mathrm{lct}(S) \geq 1$ *if* $S \neq S_{10}, S_{15}, S_{6,8}, S_{2n,2n}$. *Hence, these 58 surfaces does not contain any* $(-K)$*-polar cylinder.*
- *The surfaces,* S_{10}*,* S_{15}*,* $S_{6,8}$ *also does not contain any* $(-K)$*-polar cylinder.*
- $S_{2n,2n} \subset \mathbb{P}(1,1,n,n,2n-1)$ *and* $n = 1$ *contains a* $(-K)$*-polar cylinder. A smooth del Pezzo surface of degree* 4

Diagram for $S_{2n,2n}$



Blow ups of $\mathbb{P}(1,1,n)$

$\phi_m : B_n^m \to \mathbb{P}(1,1,n)$: blow-up along m general points .
$n = 1$ case :
B_1^m is smooth del Pezzo surfaces of degree $9 - m$ that we know the all K-stability :
B_1^m is K-polystable if and only if m is neither 1 nor 2.

Blow ups of $\mathbb{P}(1, 1, n)$

$\phi_m : B_n^m \to \mathbb{P}(1, 1, n)$: blow-up along m general points .
$n = 2$ case :
B_2^m is singular del Pezzo surfaces of degree $8 - m$ that have a sigularity of type $\mathbb{A}_1$
B_1^m is K-polystable if and only if m is neither 1 nor 2.

Theorem 3.9 (Y. Odaka, C. Spotti, S. Sun and Y. Liu and E. Denisova)

For $n = 2$, the surface B_2^m is K-polystable if and only if $m \in \{5, 6, 7\}$.

$n = 3$ case :
B_3^m is singular del Pezzo surfaces of degree $\frac{25}{3} - m > 0$ that have a sigularity of type $\mathbb{A}_2$

Blow ups of $\mathbb{P}(1,1,n)$

$\phi_m : B_n^m \to \mathbb{P}(1,1,n)$: blow-up along m general points .
$n > 3$ case :
B_n^m is singular del Pezzo surfaces of degree $\frac{(n+2)^2}{n} - m > 0$ that have a sigularity of type $\mathbb{A}_{n-1}$

Theorem 3.10 (Kim, W-)

For $n > 3$, the surface B_n^m is K-polystable if and only if $m = n + 4$.