Homomorphisms from $\mathrm{SL}(2,k)$ to $\mathrm{SL}(4,k)$ in positive characteristic

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March 5, 2025

§1. Introduction

As a continuation of our study of exponential matrices (and the Weitzenböck problem), we are interested in representations of $\mathrm{SL}(2,k)$ in positive characteristic. Representations of $\mathrm{SL}(2,k)$ behave differently whether or not the characteristic of k is zero. We begin with recalling such phenomena. We employ the following usual notations:

$$\begin{cases} k & : \text{ an algebraically closed field of characteristic } p \geq 0 \\ \mathbb{G}_a & : \text{ the additive group of } k\text{, i.e., } \mathbb{G}_a = (k, +) \\ \mathbb{G}_m & : \text{ the multiplicative group of } k\text{, i.e., } \mathbb{G}_m = (k \backslash \{0\}, \times) \end{cases}$$

Known results

- If $p = 0$, any representation $\rho : \mathrm{SL}(2,k) \to \mathrm{GL}(n,k)$ is completely reducible, i.e.,

$$\rho = \rho_1 \oplus \cdots \oplus \rho_r,$$

where each ρ_i is an irreducible representation of $\mathrm{SL}(2,k)$.

- If $p > 0$, not every representation $\rho : \mathrm{SL}(2,k) \to \mathrm{GL}(n,k)$ is completely reducible. For example, if $p = 2$, the following is not completely reducible:

$$\rho : \mathrm{SL}(2,k) \to \mathrm{GL}(3,k), \quad \rho \begin{pmatrix} a & b \\ c & d \end{pmatrix} = \left(\begin{array}{cc|c} a^2 & b^2 & a\,b \\ c^2 & d^2 & c\,d \\ \hline 0 & 0 & 1 \end{array}\right).$$

So, we cannnot understand representations of $\mathrm{SL}(2, k)$ in positive characteristic as far as we only study irreducible representations of $\mathrm{SL}(2, k)$.

Present situations.

- I have not yet found general theory of representations of $\mathrm{SL}(2, k)$ in positive characteristic.
  Tilting theory can treat tilting $\mathrm{SL}(2, k)$-modules, only.
- What is worse (or better?), I have not yet found abundant p-pathological examples of representations of $\mathrm{SL}(2, k)$ in positive characteristic.

Why we study representations of $\mathrm{SL}(2,k)$ in positive characteristic?

- Can we classify smooth projective $\mathrm{SL}(2,k)$-threefolds in positive characteristic?
- Can we classify $\mathrm{SL}(2,k)$-actions on $\mathbb{P}^3$ in positive characteristic?

Toward answering to the above questions, it would be helpful to describe homomorphisms $\mathrm{SL}(2,k) \to \mathrm{GL}(4,k)$.

- We have the following one-to-one correspondence:

$$\mathrm{Hom}(\mathrm{SL}(2,k), \mathrm{GL}(n,k)) \xleftarrow{\cong} \mathrm{Hom}(\mathrm{SL}(2,k), \mathrm{SL}(n,k))$$

$$\iota_{\mathrm{SL}(n,k)} \circ \sigma \longleftarrow\!\!\!\mid \sigma$$

Now, our interest lies in the following problem:

Problem 1

Assume $p > 0$. Describe homomorphisms

$$\mathrm{SL}(2,k) \to \mathrm{SL}(n,k).$$

For $1 \leq n \leq 3$, we already have the answer.

In this talk, we give an answer to Problem 1 for $n = 4$.

§2. Reducing Problem 1

From now on, we assume that the characteristic p of k is positive.

In this section, we shall reduce Problem 1.

(2.1) Subgroups of $\mathrm{SL}(2,k)$.

We consider the Borel subgroup of $\mathrm{SL}(2,k)$ defined by

$$\mathrm{B}(2,k) := \left\{ \begin{pmatrix} a & b \\ c & d \end{pmatrix} \in \mathrm{SL}(2,k) \;\middle|\; c = 0 \right\}.$$

We can identify the Borel subgroup $\mathrm{B}(2,k)$ with a semi-direct product $\mathbb{G}_a \rtimes \mathbb{G}_m$, where the product of elements (t,u), (t',u') of $\mathbb{G}_a \rtimes \mathbb{G}_m$ is given by

$$(t,\, u) \cdot (t',\, u') := (t + u^2\, t',\, u\, u').$$

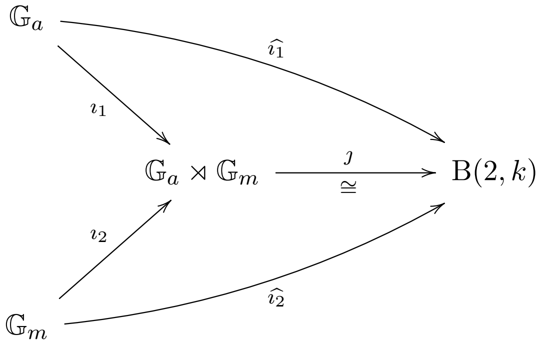
The identification is given by the following isomorphism:

$$\jmath : \mathbb{G}_a \rtimes \mathbb{G}_m \to \mathrm{B}(2,k) \qquad \jmath(t,u) := \begin{pmatrix} u & t\,u^{-1} \\ 0 & u^{-1} \end{pmatrix}.$$

Let

$$\begin{cases} \imath_1 : \mathbb{G}_a \to \mathbb{G}_a \rtimes \mathbb{G}_m & \imath_1(t) := (t,1), \\ \imath_2 : \mathbb{G}_m \to \mathbb{G}_a \rtimes \mathbb{G}_m & \imath_2(u) := (0,u), \\ \widehat{\imath_1} : \mathbb{G}_a \to \mathrm{B}(2,k) & \widehat{\imath_1} := \jmath \circ \imath_1, \\ \widehat{\imath_2} : \mathbb{G}_m \to \mathrm{B}(2,k) & \widehat{\imath_2} := \jmath \circ \imath_2. \end{cases}$$

We have the following commutative diagram:



(2.2) Homomorphisms from $\mathrm{SL}(2,k)$ to $\mathrm{SL}(n,k)$

Let $\sigma : \mathrm{SL}(2,k) \to \mathrm{SL}(n,k)$ be a morphism.

We can define the following morphisms:

$$\begin{cases} \varphi_\sigma : \mathbb{G}_a \to \mathrm{SL}(n,k) & \varphi_\sigma(t) := \sigma \begin{pmatrix} 1 & t \\ 0 & 1 \end{pmatrix}, \\ \omega_\sigma : \mathbb{G}_m \to \mathrm{SL}(n,k) & \omega_\sigma(u) := \sigma \begin{pmatrix} u & 0 \\ 0 & u^{-1} \end{pmatrix}, \\ \varphi_\sigma^- : \mathbb{G}_a \to \mathrm{SL}(n,k) & \varphi_\sigma^-(s) := \sigma \begin{pmatrix} 1 & 0 \\ s & 1 \end{pmatrix}. \end{cases}$$

Clearly, if $\sigma : \mathrm{SL}(2,k) \to \mathrm{SL}(n,k)$ is a homomorphism, the morphisms φ_σ, ω_σ, φ_σ^- are homomorphisms.

(2.3) $\omega_\sigma : \mathbb{G}_m \to \mathrm{SL}(n,k)$

For all integers d_i $(1 \leq i \leq n)$ satisfying $\sum_{i=1}^n d_i = 0$, we can define a homomorphism $\omega_{d_1,\ldots,d_n} : \mathbb{G}_m \to \mathrm{SL}(n,k)$ as

$$\omega_{d_1,\ldots,d_n}(u) := \mathrm{diag}(\, u^{d_1}, \ldots, u^{d_n}\,).$$

Let

$$\Omega(n) := \left\{ \omega_{d_1,\ldots,d_n} \;\middle|\; \begin{array}{l} d_1 \geq \cdots \geq d_n, \\ d_i = -d_{n-i+1} \quad (\, 1 \leq i \leq n \,) \end{array} \right\}.$$

For any regular matrix P of $\mathrm{GL}(n,k)$, we can define an isomorphism $\mathrm{Inn}_P : \mathrm{SL}(n,k) \to \mathrm{SL}(n,k)$ as

$$\mathrm{Inn}_P(A) := P^{-1} A P.$$

Lemma 2

For any homomorphism $\sigma : \mathrm{SL}(2,k) \to \mathrm{SL}(n,k)$, there exists a regular matrix P of $\mathrm{GL}(n,k)$ such that

$$\mathrm{Inn}_P \circ \omega_\sigma \in \Omega(n).$$

A homomorphism $\sigma : \mathrm{SL}(2,k) \to \mathrm{SL}(n,k)$ is said to be *anti-symmetric* if $\omega_\sigma \in \Omega(n)$.

$$\begin{array}{ccc} \mathrm{SL}(2,k) & \xrightarrow{\sigma} & \mathrm{SL}(n,k) \\ & \searrow^{\text{anti-symmetric}} & \downarrow \mathrm{Inn}_P \\ & & \mathrm{SL}(n,k) \end{array}$$

So, we have the following commutative diagram:

$$\begin{array}{ccc} \mathrm{Hom}(\mathrm{SL}(2,k), \mathrm{SL}(n,k)) & \longrightarrow & \mathrm{Hom}(\mathrm{SL}(2,k), \mathrm{SL}(n,k))/\sim \\ \uparrow & & \| \\ \mathrm{Hom}^a(\mathrm{SL}(2,k), \mathrm{SL}(n,k)) & \longrightarrow & \mathrm{Hom}^a(\mathrm{SL}(2,k), \mathrm{SL}(n,k))/\sim \end{array}$$

We can reduce Problem 1 to the following problem:

Problem 3

Describe anti-symmetric homomorphisms from $\mathrm{SL}(2,k)$ *to* $\mathrm{SL}(n,k)$*, i.e., describe*

$$\mathrm{Hom}^a(\mathrm{SL}(2,k),\mathrm{SL}(n,k))/\sim.$$

In Section 3, we meet the following commutative diagram:

$$\begin{array}{ccc}
\mathrm{Hom}(\mathrm{SL}(2,k),\mathrm{SL}(n,k)) & \longrightarrow & \mathrm{Hom}(\mathrm{B}(2,k),\mathrm{SL}(n,k)) \\
\uparrow & & \uparrow \\
\mathrm{Hom}^a(\mathrm{SL}(2,k),\mathrm{SL}(n,k)) & \longrightarrow & \mathrm{Hom}^a(\mathrm{B}(2,k),\mathrm{SL}(n,k))
\end{array}$$

§3. Homomorphisms $\mathrm{B}(2,k) \to \mathrm{SL}(n,k)$

(3.1) φ_ψ and ω_ψ

Given a morphism $\psi : \mathrm{B}(2,k) \to \mathrm{SL}(n,k)$, we can define the following morphisms:

$$\begin{cases} \varphi_\psi : \mathbb{G}_a \to \mathrm{SL}(n,k) & \varphi_\psi(t) := \psi \begin{pmatrix} 1 & t \\ 0 & 1 \end{pmatrix}, \\ \omega_\psi : \mathbb{G}_m \to \mathrm{SL}(n,k) & \omega_\psi(u) := \psi \begin{pmatrix} u & 0 \\ 0 & u^{-1} \end{pmatrix}. \end{cases}$$

If ψ is a homomorphism, then φ_ψ and ω_ψ are homomorphisms.

A homomorphism $\psi : \mathrm{B}(2,k) \to \mathrm{SL}(n,k)$ is said to be *anti-symmetric* if $\omega_\psi \in \Omega(n)$.

Lemma 4

The following diagram is commutative:

$$\begin{array}{ccc}
\mathrm{Hom}(\mathrm{SL}(2,k),\mathrm{SL}(n,k)) & \longrightarrow & \mathrm{Hom}(\mathrm{B}(2,k),\mathrm{SL}(n,k)) \\
\uparrow & & \uparrow \\
\mathrm{Hom}^a(\mathrm{SL}(2,k),\mathrm{SL}(n,k)) & \longrightarrow & \mathrm{Hom}^a(\mathrm{B}(2,k),\mathrm{SL}(n,k)) \\
\downarrow & & \downarrow \\
\mathrm{Hom}^a(\mathrm{SL}(2,k),\mathrm{SL}(n,k))/\sim & \longrightarrow & \mathrm{Hom}^a(\mathrm{B}(2,k),\mathrm{SL}(n,k))/\sim
\end{array}$$

(3.2) $\psi_{\varphi,\omega}$

Given morphisms $\varphi : \mathbb{G}_a \to \mathrm{SL}(n,k)$ and $\omega : \mathbb{G}_m \to \mathrm{SL}(n,k)$, we can define a morphism $\psi_{\varphi,\omega} : \mathbb{G}_a \rtimes \mathbb{G}_m \to \mathrm{SL}(n,k)$ as

$$\psi_{\varphi,\omega}(t,u) := \varphi(t) \cdot \omega(u).$$

Lemma 5

Let $\psi : \mathrm{B}(2,k) \to \mathrm{SL}(n,k)$ be a homomorphism. Then the following assertions (1) and (2) hold true:

(1) $\psi \circ \jmath = \psi_{\varphi_\psi,\omega_\psi}$.

(2) *There exist unique homomorphisms $\varphi : \mathbb{G}_a \to \mathrm{SL}(n,k)$ and $\omega : \mathbb{G}_m \to \mathrm{SL}(n,k)$ such that $\psi \circ \jmath = \psi_{\varphi,\omega}$.*

Lemma 6

Let $\varphi : \mathbb{G}_a \to \mathrm{SL}(n,k)$ *and* $\omega : \mathbb{G}_m \to \mathrm{SL}(n,k)$ *be morphisms. Then the morphism* $\psi_{\varphi,\omega} : \mathbb{G}_a \rtimes \mathbb{G}_m \to \mathrm{SL}(n,k)$ *is a homomorphism if and only if the following conditions* (1), (2), (3) *hold true:*

(1) φ *is a homomorphism.*

(2) ω *is a homomorphism.*

(3) $\omega(u)\,\varphi(t)\,\omega(u)^{-1} = \varphi(u^2\,t)$ *for all* $(t,u) \in \mathbb{G}_a \rtimes \mathbb{G}_m$.

(3.3) Anti-symmetric homomorphisms $\psi : \mathrm{B}(2,k) \to \mathrm{SL}(n,k)$

Recall $\mathrm{B}(2,k) = \langle\, \mathbb{G}_a^+,\ \mathbb{G}_m \,\rangle$.

Lemma 7

Let $\psi : \mathrm{B}(2,k) \to \mathrm{SL}(n,k)$ be an anti-symmetric homomorphism. Then the homomorphism $\varphi_\psi : \mathbb{G}_a \to \mathrm{SL}(n,k)$ satisfies the following condition:

For any $t \in \mathbb{G}_a$, the regular matrix $\varphi_\psi(t)$ is upper triangular.

We obtain the forms of homomorphisms $\varphi : \mathbb{G}_a \to \mathrm{SL}(n,k)$ satisfying $1 \leq n \leq 5$ and the following condition:

For any $t \in \mathbb{G}_a$, the regular matrix $\varphi(t)$ is upper triangular.

§4. $\mathrm{Hom}^a(\mathrm{SL}(2,k),\mathrm{SL}(4,k))$

We can choose 26 pairs (φ^*,ω^*) from $\mathrm{Hom}^a(\mathrm{B}(2,k),\mathrm{SL}(4,k))$.

We label the types as (I), (II) – (XXVI).

(I) Assume $p \geq 5$. Let $e_1 \geq 0$. Let $d_1 := 3\,p^{e_1}$ and $d_2 := p^{e_1}$. Let $\omega^* : \mathbb{G}_m \to \mathrm{SL}(4,k)$ be the homomorphism defined by

$$\omega^*(u) := \mathrm{diag}(\,u^{d_1},\ u^{d_2},\ u^{-d_2},\ u^{-d_1}\,).$$

Let $\varphi^* : \mathbb{G}_a \to \mathrm{SL}(4,k)$ be the homomorphism defined by

$$\varphi^*(t) := \begin{pmatrix} 1 & t^{p^{e_1}} & \frac{1}{2}\,t^{2\,p^{e_1}} & \frac{1}{6}\,t^{3\,p^{e_1}} \\ 0 & 1 & t^{p^{e_1}} & \frac{1}{2}\,t^{2\,p^{e_1}} \\ 0 & 0 & 1 & t^{p^{e_1}} \\ 0 & 0 & 0 & 1 \end{pmatrix}.$$

(II) Assume $p = 3$. Let $e_1 \geq 0$. Let $d_1 = p^{e_1+1}$ and $d_2 = p^{e_1}$. Let $\omega^* : \mathbb{G}_m \to \mathrm{SL}(4,k)$ be the homomorphism defined by

$$\omega^*(u) := \mathrm{diag}(\, u^{d_1},\; u^{d_2},\; u^{-d_2},\; u^{-d_1}\,).$$

Let $\varphi^* : \mathbb{G}_a \to \mathrm{SL}(4,k)$ be the homomorphism defined by

$$\varphi^*(t) := \begin{pmatrix} 1 & t^{p^{e_1}} & \frac{1}{2}\, t^{2\,p^{e_1}} & t^{p^{e_1+1}} \\ 0 & 1 & t^{p^{e_1}} & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \end{pmatrix}.$$

(III) Assume $p \geq 3$. Let $e_1 \geq 0$. Let $d_1 = 3\,p^{e_1}$ and $d_2 = p^{e_1}$. Let $\omega^* : \mathbb{G}_m \to \mathrm{SL}(4,k)$ be the homomorphism defined by

$$\omega^*(u) := \mathrm{diag}(\,u^{d_1},\ u^{d_2},\ u^{-d_2},\ u^{-d_1}\,).$$

Let $\varphi^* : \mathbb{G}_a \to \mathrm{SL}(4,k)$ be the homomorphism defined by

$$\varphi^*(t) := \begin{pmatrix} 1 & t^{p^{e_1}} & \frac{1}{2}\,t^{2\,p^{e_1}} & 0 \\ 0 & 1 & t^{p^{e_1}} & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \end{pmatrix}.$$

$\vdots$

(XXVI)

Theorem 8

The following assertions (1) *and* (2) *hold true:*

(1) *Let* (φ^*, ω^*) *be a pair of the form* (ν), *where* $\nu = \mathrm{I}, \ldots, \mathrm{XXVI}$. *Then the morphism* $\psi_{\varphi^*, \omega^*} : \mathbb{G}_a \rtimes \mathbb{G}_m \to \mathrm{SL}(4,k)$ *is a homomorphism and* $\omega^* \in \Omega(4)$.

(2) *Let* $\psi : \mathrm{B}(2,k) \to \mathrm{SL}(4,k)$ *be an antisymmetric homomorphism. Express* ψ *as* $\psi \circ \jmath = \psi_{\varphi, \omega}$ *for some* (φ, ω) *of* $\mathrm{Hom}(\mathbb{G}_a, \mathrm{SL}(4,k)) \times \Omega(4)$. *Then there exists an element* (φ^*, ω^*) *of* $\mathrm{Hom}(\mathbb{G}_a, \mathrm{SL}(4,k)) \times \Omega(4)$ *such that the following conditions* (a) *and* (b) *hold true:*

   (a) $(\varphi, \omega) \sim (\varphi^*, \omega^*)$.

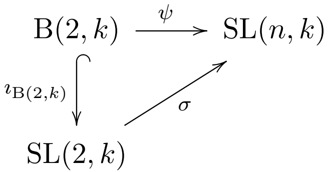
   (b) (φ^*, ω^*) *has one of the above forms* (I) – (XXVI).

Recall the following commutative diagram:

$$
\begin{array}{ccc}
\mathrm{Hom}(\mathrm{SL}(2,k),\mathrm{SL}(4,k)) & \longrightarrow & \mathrm{Hom}(\mathrm{B}(2,k),\mathrm{SL}(4,k)) \\
\uparrow & & \uparrow \\
\mathrm{Hom}^a(\mathrm{SL}(2,k),\mathrm{SL}(4,k)) & \longrightarrow & \mathrm{Hom}^a(\mathrm{B}(2,k),\mathrm{SL}(4,k)) \\
\downarrow & & \downarrow \\
\mathrm{Hom}^a(\mathrm{SL}(2,k),\mathrm{SL}(4,k))/\sim & \longrightarrow & \mathrm{Hom}^a(\mathrm{B}(2,k),\mathrm{SL}(4,k))/\sim
\end{array}
$$

$\mathrm{Hom}^a(\mathrm{B}(2,k),\mathrm{SL}(4,k))/\sim$ has at most 26 types.

Let $\imath_{\mathrm{B}(2,k)} : \mathrm{B}(2,k) \to \mathrm{SL}(2,k)$ be the inclusion homomorphism. A homomorphism $\psi : \mathrm{B}(2,k) \to \mathrm{SL}(n,k)$ is said to be *extendable* if there exists a homomorphism $\sigma : \mathrm{SL}(2,k) \to \mathrm{SL}(n,k)$ such that $\sigma \circ \imath_{\mathrm{B}(2,k)} = \psi$, i.e., the following diagram is commutative:



Lemma 9

Let (φ^*, ω^*) *be of the form* (I). *Then* $\psi_{(\varphi^*,\omega^*)}$ *is uniquely extendable to the following* $\sigma^* : \mathrm{SL}(2,k) \to \mathrm{SL}(4,k)$*:*

$$\sigma^* \begin{pmatrix} a & b \\ c & d \end{pmatrix}$$

$$:= \begin{pmatrix} a^{3\,p^{e_1}} & a^{2\,p^{e_1}}\, b^{p^{e_1}} & \frac{1}{2}\, a^{p^{e_1}}\, b^{2\,p^{e_1}} & \frac{1}{6}\, b^{3\,p^{e_1}} \\ 3\,a^{2\,p^{e_1}}\, c^{p^{e_1}} & a^{p^{e_1}} \cdot (a\,d + 2\,b\,c)^{p^{e_1}} & b^{p^{e_1}} \cdot (a\,d + \frac{1}{2}\,b\,c)^{p^{e_1}} & \frac{1}{2}\, b^{2\,p^{e_1}}\, d^{p^{e_1}} \\ 6\,a^{p^{e_1}}\, c^{2\,p^{e_1}} & 4\,c^{p^{e_1}} \cdot (a\,d + \frac{1}{2}\,b\,c)^{p^{e_1}} & d^{p^{e_1}} \cdot (a\,d + 2\,b\,c)^{p^{e_1}} & b^{p^{e_1}}\, d^{2\,p^{e_1}} \\ 6\,c^{3\,p^{e_1}} & 6\,c^{2\,p^{e_1}}\, d^{p^{e_1}} & 3\,c^{p^{e_1}}\, d^{2\,p^{e_1}} & d^{3\,p^{e_1}} \end{pmatrix}.$$

This homomorphism σ^* is well known.

Lemma 10

Let (φ^*, ω^*) *be of the form* (II). *Then* $\psi_{(\varphi^*,\omega^*)}$ *is uniquely extendable to the following* $\sigma^* : \mathrm{SL}(2,k) \to \mathrm{SL}(4,k)$*:*

$$\sigma^* \begin{pmatrix} a & b \\ c & d \end{pmatrix} = \begin{pmatrix} a^{p^{e_1}+1} & a^{2\,p^{e_1}}\, b^{p^{e_1}} & \frac{1}{2}\, a^{p^{e_1}}\, b^{2\,p^{e_1}} & b^{p^{e_1}+1} \\ 0 & a^{p^{e_1}} & b^{p^{e_1}} & 0 \\ 0 & c^{p^{e_1}} & d^{p^{e_1}} & 0 \\ c^{p^{e_1}+1} & c^{2\,p^{e_1}}\, d^{p^{e_1}} & \frac{1}{2}\, c^{p^{e_1}}\, d^{2\,p^{e_1}} & d^{p^{e_1}+1} \end{pmatrix}.$$

Lemma 11

Let (φ^*, ω^*) *be of the form* (III). *Then* $\psi_{(\varphi^*,\omega^*)}$ *is not extendable.*

§5. $\mathrm{Hom}^a(\mathrm{SL}(2,k),\mathrm{SL}(4,k))/\sim$

Theorem 12

Let $\sigma : \mathrm{SL}(2,k) \to \mathrm{SL}(4,k)$ be a homomorphism. Then there exists a homomorphism $\sigma^* : \mathrm{SL}(2,k) \to \mathrm{SL}(4,k)$ satisfying the following conditions (i) *and* (ii)*:*

(i) *σ and σ^* are equivalent, i.e., $\sigma \sim \sigma^*$.*

(ii) *σ^* has one of the forms* $(\mathrm{I})^*$, $(\mathrm{II})^*$, $(\mathrm{IV})^*$, $(\mathrm{V})^*$, $(\mathrm{VII})^*$, $(\mathrm{IX})^*$, $(\mathrm{XI})^*$, $(\mathrm{XV})^*$, $(\mathrm{XIX})^*$, $(\mathrm{XXIV})^*$, $(\mathrm{XXVI})^*$.

$p=2$	$p=3$	$p\geq 5$	$p\geq 2$	d
		(I)*		$(0,0)$
	(II)*			$(0,0)$
(IV)*	(IV)*	(IV)*	(IV)*	$(0,0)$
(V)*				$(1,1)$
	(VII)*			$(0,0)$
	(IX)*	(IX)*		$(1,1)$
(XI)*				$(1,2)$
(XV)*	(XV)*	(XV)*	(XV)*	$(0,0)$
(XIX)*				$(2,1)$
(XXIV)*	(XXIV)*	(XXIV)*	(XXIV)*	$(2,2)$
(XXVI)*	(XXVI)*	(XXVI)*	(XXVI)*	$(4,4)$
7 types	7 types	6 types	4 types	

The following might be a new representation of $\mathrm{SL}(2,k)$.

Assume $p=2$. For all integer $e_1 \geq 0$, we can define a homomorphism $\sigma_{(\mathrm{V})^\sharp,\, e_1} : \mathrm{SL}(2,k) \to \mathrm{SL}(4,k)$ as

$$\sigma_{(\mathrm{V})^\sharp,\, e_1}\begin{pmatrix} a & b \\ c & d \end{pmatrix} := \begin{pmatrix} 1 & 0 & 0 & 0 \\ a^{p^{e_1}}\, b^{p^{e_1}} & a^{p^{e_1+1}} & b^{p^{e_1+1}} & 0 \\ c^{p^{e_1}}\, d^{p^{e_1}} & c^{p^{e_1+1}} & d^{p^{e_1+1}} & 0 \\ b^{p^{e_1}}\, c^{p^{e_1}} & a^{p^{e_1}}\, c^{p^{e_1}} & b^{p^{e_1}}\, d^{p^{e_1}} & 1 \end{pmatrix}.$$

Lemma 13

Let $e_1 \geq 0$, let σ^* be the homomorphism given in $(\mathrm{V})^*$ and let $P := P_{3,4}\, P_{1,2} \in \mathrm{GL}(4,k)$. Then $\mathrm{Inn}_P \circ \sigma^* = \sigma_{(\mathrm{V})^\sharp,\, e_1}$.

Thank you very much for your attention.