Completion of the affine 3-space into sextic del Pezzo fibrations

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Contents

- This talk is based on j.w.w Adrien Dubouloz, Takashi Kishimoto (arXiv:2401.02857)

§1 Backgrounds

Setting

- $k = \bar{k}$: field of characteristic 0
- **Completion** of the affine space $\mathbb{A}^n$
  := proj. (connected) variety containing $\mathbb{A}^n$ as a dense open subscheme
- For completion X of $\mathbb{A}^n$, $B = X \backslash \mathbb{A}^n$ is called the **boundary divisor**
- e.g. $\mathbb{P}^n$ is a smooth completion of $\mathbb{A}^n$,
  $B = \mathbb{P}^n \backslash \mathbb{A}^n$ is a hyperplane section
- **Aim: classify sm. proj. completions X of $\mathbb{A}^n$ and its boundary B**

Known results

- $\rho(X) :=$ Picard rank of X
- $n = 1$: $(X, B) \cong (\mathbb{P}^1, \mathrm{pt})$
- $n = 2,\ \rho(X) = 1$: $(X, B) \cong (\mathbb{P}^2,$ line$)$
- $n = 2,\ \rho(X) = 2$: $X =$ Hirzebruch surface, (X, B) is classified [Mori73] using birat. transform. preserving $\mathbb{A}^2$

Known results

- $n = 3$, $\rho(X) = 1$: $X \cong \mathbb{P}^3$, quadric $\mathbb{Q}^3 \subset \mathbb{P}^4$, quintic del Pezzo 3-fold V_5, or Mukai 3-fold of $g = 12$. (X, B) is classified [Peternell, Schneider, Prokhorov, Mukai, Furushima, etc.]
- $n = 3$, $\rho(X) = 2$: (X, B) is NOT classified, few examples ←**Today** [Müller-Stach, Kishimoto, N., Huang-Montero, etc.]

§2 Results of [DKN]

MMP strategy

- **In what follows, we suppose $n = 3$.**

- How to treat completion X of $\mathbb{A}^3$ w/ boundary B in the case where $\rho(X) \geq 2$?
  $\cdots$ run an MMP $\varphi : X \dashrightarrow X_1 \dashrightarrow \cdots \dashrightarrow X_{\min}$ and classify $(X_{\min}, \varphi_* B)$ instead

- Disadvantage : $X_{\min}$ has at worst $\mathbb{Q}$-factorial terminal singularities
  φ does not preserve $X \backslash B \cong \mathbb{A}^3$ in general

- Advantage : $X_{\min}$ has a Mori fiber space structure

- Question : How many Mori fiber spaces with $\mathbb{Q}$-fact. term. sing. contain $\mathbb{A}^3$?

Del Pezzo fibrations

- **Del Pezzo fibration** (in this talk) := proj. 3-fold X with $\mathbb{Q}$-fact. term. sing. endowed with an extremal contraction $\pi\colon X \to C$ to a curve
- A general π-fiber S is a sm. **del Pezzo surface** ($\iff -K_S$ is ample)
- d : **degree of del Pezzo fibration** $:= (-K_S)^2 \in \{1,2,\cdots,9\}$
  ($S = \mathbb{P}^1 \times \mathbb{P}^1$ or the blow-up of $\mathbb{P}^2$ at $(9-d)$ points in general position)
- $S \neq \mathrm{Bl}_{1pt}\mathbb{P}^2, \neq \mathrm{Bl}_{2pts}\mathbb{P}^2$. ($\because$ relative Picard rank =1) In particular, $d \neq 7$.

Theorem A

- Question A: which smooth del Pezzo surface can appear as a closed fiber of a del Pezzo fibration $\pi: X \to C \ (\cong \mathbb{P}^1)$ whose total space is a completion of $\mathbb{A}^3$?

- **Theorem A [DKN24]**

  Let S : sm. del Pezzo surface of degree d, $\neq \mathrm{Bl}_{1pt}\mathbb{P}^2$, $\neq \mathrm{Bl}_{2pts}\mathbb{P}^2$

  Then $\exists \pi: X \to \mathbb{P}^1$: del Pezzo fibration of degree d,

  $\exists B_f$: π-fiber, $\exists B_h$: prime divisor on X

  s.t ・S is isomorphic to some π-fiber ・$X \backslash (B_f \cup B_h) \cong \mathbb{A}^3$

Horizontal divisor

$\pi\colon X \to \mathbb{P}^1$: del Pezzo fibration of degree d,
B_f : π-fiber, B_h : prime divisor on X
s.t ・$X \backslash (B_f \cup B_h) \cong \mathbb{A}^3$

- In the situation of Theorem A, the induced morphism $\bar{\pi}\colon B_h \to \mathbb{P}^1$ is surjective
- $\mathrm{Pic}(\mathbb{A}^3) = 0$ & $\Gamma(\mathbb{A}^3)^* = k^* \Rightarrow \{B_f, B_h\}$ is a $\mathbb{Z}$-basis of $\mathrm{Cl}(X)$
- If $d \leq 6$, then $\{B_f, -K_X\}$ is also a $\mathbb{Z}$-basis $\Rightarrow B_h \sim_{\mathbb{P}^1} -K_X$

  $\Rightarrow \bar{\pi}$: plane cubic curve fibration
- Question B: when $d \leq 6$, which plane cubic curve can appear as a closed $\bar{\pi}$-fiber?

Theorem B

- Question B : when $d \leq 6$, which plane cubic curve can appear as a closed $\overline{\pi}$-fiber?

- **Theorem B [DKN24]**
  Let C : integral plane cubic curve, $1 \leq d \leq 6$

  Then $\exists \pi \colon X \to \mathbb{P}^1$: del Pezzo fibration of degree d,

  $\exists B_f$: π-fiber, $\exists B_h$: prime divisor on X

  s.t ・C is isomorphic to some $\overline{\pi}$-fiber ・$X \backslash (B_f \cup B_h) \cong \mathbb{A}^3$

- Main topic of this talk = Theorem B in the case where $d = 6$

§3 General case: $d \neq 6$

General framework

- How to get completion of $\mathbb{A}^3$ into del Pezzo fibrations?

Step 1 : Find suitable pencils of del Pezzo surfaces on known completions of $\mathbb{A}^3$ into $\mathbb{Q}$-Fano 3-folds

Step 2 : Take appropriate resolution of indeterminacy of pencils

Step 3 : Run relative MMPs

General framework

Step 1 : Find suitable pencils of del Pezzo surfaces on known completions of $\mathbb{A}^3$ into $\mathbb{Q}$-Fano 3-folds

- Definition: **An H-special del Pezzo pencil** is a triple (X, H, ψ) s.t.

  X : proj. 3-fold of divisor class rank one with ($\mathbb{Q}$-fact.) term. sing.

  H : an effective prime Weil divisor on X s.t. $\mathrm{Cl}(X) = \mathbb{Z}[H]$

  $\psi: X \dashrightarrow \mathbb{P}^1$: pencil of (Cartier) divisors satisfying the following:

  (a) ψ has a member which is a sm. del Pezzo surface,

  (b) $\exists m \geq 1$ s.t. mH is a member of ψ,

  (c) The base scheme $\mathrm{Bs}(\psi)$ is irred., (d) If $m = 1$, then $\mathrm{Bs}(\psi)$ is reduced.

General framework

<u>Step 2</u> : Take appropriate resolution of indeterminacy of pencils

- Definition: the graph of a pencil of divisors $\psi \colon X \dashrightarrow \mathbb{P}^1$ is the scheme theoretic closure $\Gamma \hookrightarrow X \times \mathbb{P}^1$ of the restriction of ψ to its domain of definition. **The graph resolution** is the induced morphism $\gamma \colon \Gamma \to X$.
  E_Γ := the exceptional locus of γ

- Definition: **a thrifty resolution** of a pencil of divisors $\psi \colon X \dashrightarrow \mathbb{P}^1$ on a normal proj. var. X is a resolution $\tau \colon X' \to X$ s.t. the induced morphism $\sigma \colon X' \to \Gamma$ is a $\mathbb{Q}$-factorial terminalization of the normalization of Γ.

General framework

<u>Step 3</u> : Run relative MMPs

- **Theorem C [DKN24]**

  Let (X, H, ψ) : H-special del Pezzo pencil s.t. mH corresp. to $\psi^*(\infty)$

  Suppose that $m = 1$ or that $m \geq 2$ and $X \backslash H$ is smooth.

  Then $\forall \tau \colon Y \to X$: thrifty resolution of ψ, $\forall \varphi \colon Y \dashrightarrow \tilde{Y}$: MMP/$\mathbb{P}^1$, the output is a del Pezzo fibration $\tilde{\pi} \colon \tilde{Y} \to \mathbb{P}^1$ s.t. $\tilde{Y}$ is a completion of $X \backslash H$ with boundary divisor $B = B_h \cup B_f = \varphi_* \sigma_*^{-1} E_\Gamma \cup \varphi_*(\sigma^{-1}(\gamma_*^{-1} H))$,

  where $\sigma \colon Y \to \Gamma$ is the induced morphism.

  Moreover, $\varphi \circ \tau^{-1}$ induces $(\overline{\psi^*(c)}, (\mathrm{Bs}(\psi))_{\mathrm{red}}) \xrightarrow{\sim} (\tilde{\pi}^*(c), B_h \cap \tilde{\pi}^*(c))$ $(\forall c \in \mathbb{P}^1 \backslash \{\infty\})$

Proof of Theorems A and B ($d \neq 6$ case)

- To prove Theorem A (resp. Theorem B), It suffices to show:

- $\forall S$: sm. del Pezzo surface (resp. $\forall C$: int. plane cubic curve), $\exists H$-special del Pezzo pencil (X, H, ψ) s.t.

  ・S is a member of ψ ・$X \backslash H \cong \mathbb{A}^3$

  (resp. a sm. dP_d-surf. is a member of ψ & $(\mathrm{Bs}(\psi))_{\mathrm{red}} \cong C$)

Theorem A

S : sm. dP surf. of deg=d, $\neq \mathrm{Bl}_{1pt}\mathbb{P}^2$, $\neq \mathrm{Bl}_{2pts}\mathbb{P}^2$

$\Rightarrow \exists \pi : X \to \mathbb{P}^1$: dP_d-fib'n

$\exists B_f$: π-fiber,

$\exists B_h$: prime divisor on X

s.t ・$S \cong$ some π-fiber ・$X \backslash (B_f \cup B_h) \cong \mathbb{A}^3$

Theorem B

C : integral plane cubic curve, $1 \leq d \leq 6$

$\Rightarrow \exists \pi : X \to \mathbb{P}^1$: dP_d-fib'n

$\exists B_f$: π-fiber,

$\exists B_h$: prime divisor on X

s.t ・$C \cong$ some $\bar{\pi}$-fiber ・$X \backslash (B_f \cup B_h) \cong \mathbb{A}^3$

Proof of Theorems A and B ($d \neq 6$ case)

- S : sm. del Pezzo surface of $d \neq 6,7$

$$\Rightarrow \begin{cases} S \subset X = \mathbb{P}(1,1,2,3) : \text{sextic hypersurface} & (d=1) \\ S \subset X = \mathbb{P}(1,1,1,2) : \text{quartic hypersurface} & (d=2) \\ S \subset X = \mathbb{P}^3 : \text{cubic surface} & (d=3) \\ S \subset X = \mathbb{Q}^3 \subset \mathbb{P}^4 : \text{(2,2)-complete intersection} & (d=4) \\ S \subset X = V_5 \subset \mathbb{P}^6 : \text{hyperplane section} & (d=5) \\ S \subset X = \mathbb{P}^3 : \text{quadric surface} & (d=8) \\ S \subset X = \mathbb{P}^3 : \text{plane} & (d=9) \end{cases}$$

- $\exists H \in |\mathcal{O}_X(1)|$ s.t. $X \backslash H \cong \mathbb{A}^3$ ・$\psi :=$ pencil gen'd by S and mult. of H

Proof of Theorems A and B ($d = 5$ case)

- Sm. del Pezzo surface of $d = 5$ is unique up to isom.
- V_5 : quintic del Pezzo 3-fold V_5, $\rho(V_5) = 1$ (unique up to isom.)
- $\forall H \in |\mathcal{O}_{V_5}(1)|$ is a del Pezzo surface of $d = 5$
- $0 \to H^0(V_5, \mathcal{O}_{V_5}) \to H^0(V_5, \mathcal{O}_{V_5}(1)) \to H^0(H, \mathcal{O}_H(-K_H)) \to 0$ (exact)
- [Peternell-Schneider, Furushima-Nakayama] $\exists ! H^0$, $\exists ! H^\infty \in |\mathcal{O}_{V_5}(1)|$ up to $\mathrm{Aut}(V_5)$-action s.t. $V_5 \backslash H^0 \cong V_5 \backslash H^\infty \cong \mathbb{A}^3$, H^0 is normal, H^∞ is non-normal

Proof of Theorems A and B ($d = 5$ case)

- Lemma: (1) $\forall C$: integral plane cubic curve, $\exists D \subset H^0_{\mathrm{reg}}$ s.t. $C \cong D$

  (2) For such D, $\exists S \in |\mathcal{O}_{V_5}(1)|$: smooth member s.t. $H^0 \cap S = D$

- Take $\psi \colon X = V_5 \dashrightarrow \mathbb{P}^1$ s.t. $\overline{\psi^*(0)} = S$ and $\overline{\psi^*(\infty)} = H^0$

  $\to (X, H^0, \psi)$: H^0-special del Pezzo pencil s.t. $\mathrm{Bs}(\psi) = D \cong C$

- By Theorem C, we obtain a del Pezzo fibration $\tilde{\pi} \colon \tilde{Y} \to \mathbb{P}^1$ s.t.
  - $\tilde{Y} \backslash (B_f \cup B_h) \cong V_5 \backslash H^0 \cong \mathbb{A}^3$
  - $(\tilde{\pi}^{-1}(0), \overline{\pi}^{-1}(0) = \tilde{\pi}^{-1}(0)|_{B_h}) \cong (\psi^{-1}(0), \mathrm{Bs}(\psi)) \cong (S, D)$ $\square$

§4 Remaining case: $d = 6$

Difficulty

- We do not know an example of an H-special del Pezzo pencil (X, H, ψ) s.t. ψ has a member which is a sm. del Pezzo surface of $d = 6$
- e.g. del Pezzo variety of dim=3, $d = 6$ has divisor class rank ≥ 2

Definition: **An H-special del Pezzo pencil** is a triple (X, H, ψ) s.t.
X : proj. 3-fold of divisor class rank one with ($\mathbb{Q}$-fact.) term. sing⋯.

Proof of Theorem A ($d = 6$ case)

- Sm. del Pezzo surface of $d = 6$ is unique up to isom.

- E.g. ([Prokhorov16]).
  l : the non-normal locus of $H^\infty \in |\mathcal{O}_{V_5}(1)|$, which is a line

  $B \subset H^\infty \subset V_5$: quartic rational curve

  $f\colon Y \to V_5$: the blow-up along B with the exceptional divisor E_f

  $\Upsilon \subset Y$: the strict transform of l

- Then ・ $-K_Y$: nef & big (i.e. Y is weak Fano)

  ・ Υ is the unique $(-K_Y)$-trivial curve

Proof of Theorem A ($d = 6$ case)

- E.g. ([Prokhorov16]).
  - ∃ diagram
    $$\begin{array}{ccccc} & & Y & \overset{\chi}{\dashrightarrow} & Y^+ & & \\ & \overset{f}{\swarrow} & & & & \overset{\pi}{\searrow} & \\ V_5 & & & & & & \mathbb{P}^1 \end{array}$$
    s.t. χ : Atiyah flop of Υ
    π : del Pezzo fib'n of $d = 6$
  - $\mathbb{A}^3 \cong V_5 \backslash H^\infty \cong Y \backslash ((H^\infty)_Y \cup E_f) \cong Y^+ \backslash ((H^\infty)_{Y^+} \cup (E_f)_{Y^+}) \cong Y^+ \backslash (B_f \cup B_h)$
  - general fibers of $\overline{\pi}: B_h \to \mathbb{P}^1$ is nodal.

- Remark: Replacing H^∞ as H^0, we obtain another completion of $\mathbb{A}^3$ s.t. general fibers of $\overline{\pi}: B_h \to \mathbb{P}^1$ is cuspidal.

Proof of Theorem B ($d = 6$ case)

- We are reduced to proving:

- **Theorem B'**

  Let C : smooth plane cubic curve

  Then $\exists \pi : X \to \mathbb{P}^1$: del Pezzo fibration of degree $d = 6$,

  $\exists B_f$: π-fiber, $\exists B_h$: prime divisor on X

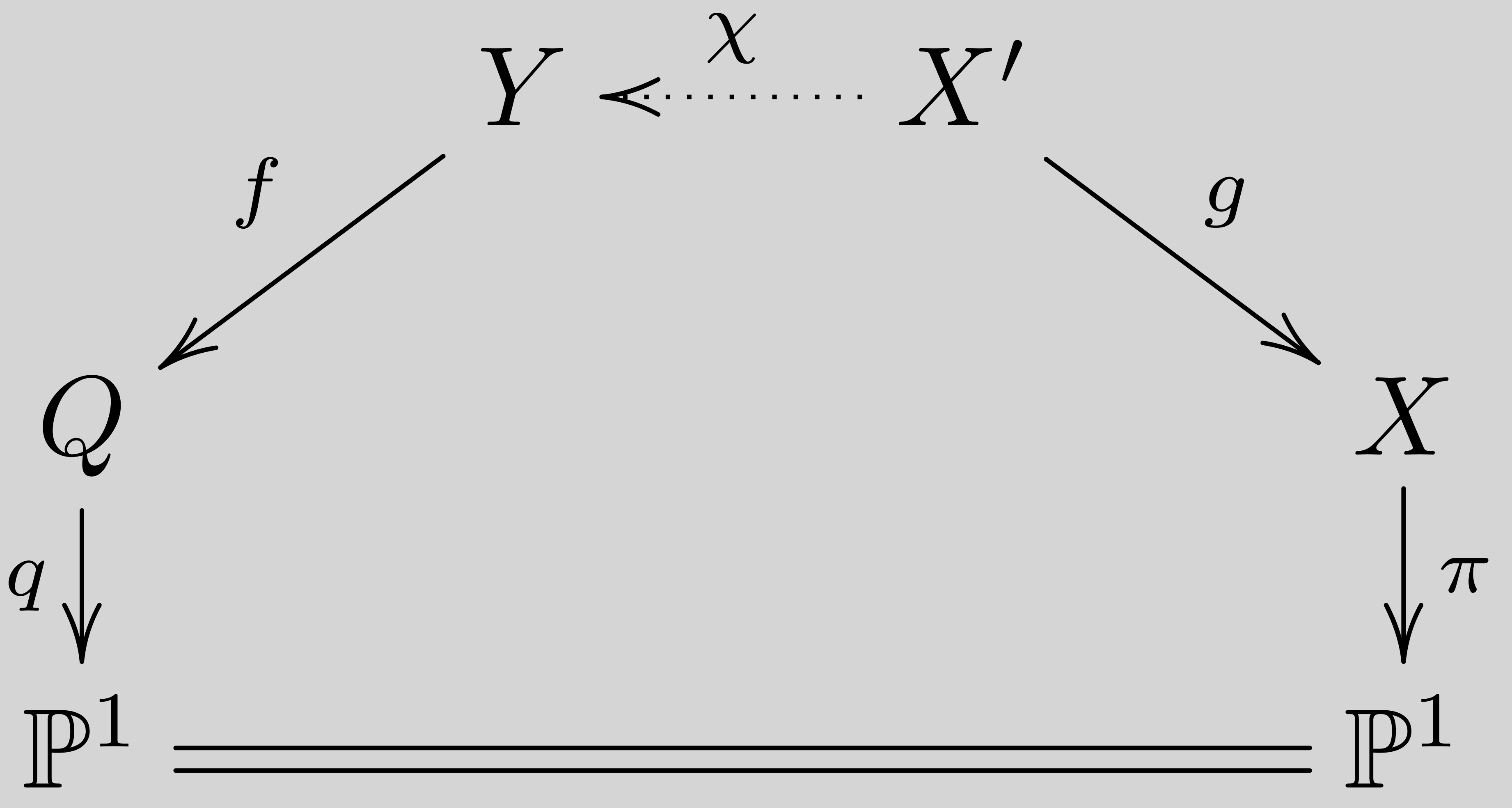
  s.t ・C is isomorphic to some $\bar{\pi}$-fiber ・$X \backslash (B_f \cup B_h) \cong \mathbb{A}^3$

Sarkisov link

C : smooth plane cubic curve
$\pi \colon X \to \mathbb{P}^1$: del Pezzo fib'n of degree $d = 6$,
B_f : π-fiber, $\exists B_h$: prime divisor on X
s.t • $C \cong$ some π-fiber • $X \backslash (B_f \cup B_h) \cong \mathbb{A}^3$

- In the situation of Theorem B',
  suppose that X is smooth.

- [Fukuoka18]: $\forall s_0$: π-section, $\exists$diagram s.t.
  g : blow-up along s_0,
  χ : the flop/$\mathbb{P}^1$ (or $\mathrm{id}_{X'}$),
  f : blow-up along a sm. q-trisection (say T)
  q : del Pezzo fib'n of degree 8



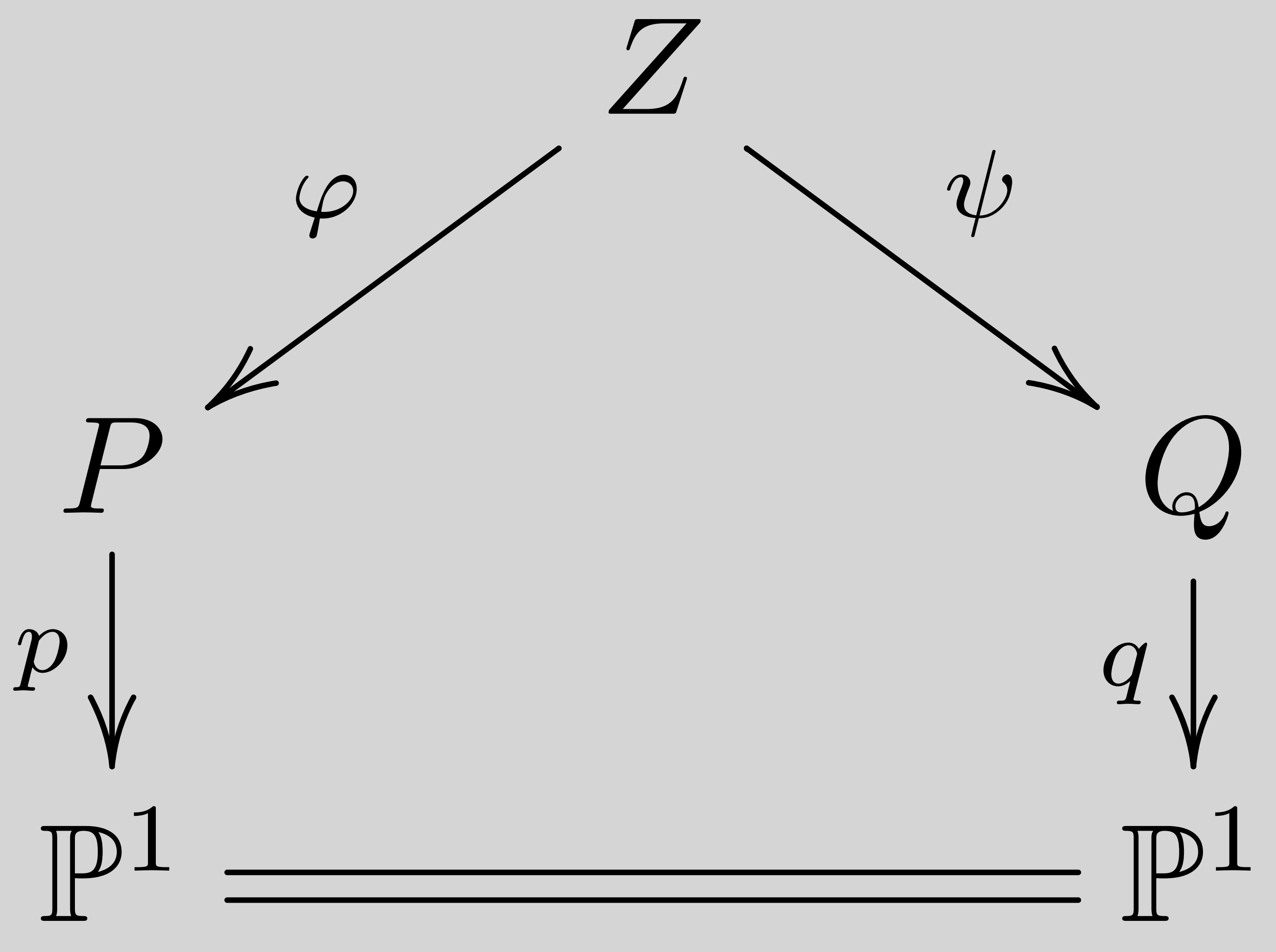
- [Fukuoka17]: $\exists$opposite construction if $-K_Y$: $(q \circ f)$-nef & $(q \circ f)$-big

Sarkisov link

> C : smooth plane cubic curve
> $\pi\colon X \to \mathbb{P}^1$: del Pezzo fib'n of degree $d = 6$,
> B_f : π-fiber, $\exists B_h$: prime divisor on X
> s.t • $C \cong$ some π-fiber • $X \setminus (B_f \cup B_h) \cong \mathbb{A}^3$

- $q\colon Q \to \mathbb{P}^1$: del Pezzo fib'n of degree 8
  Suppose that Q is smooth.

- [D'Souza88]: $\forall s$: q-section, ∃diagram s.t.
  ψ : blow-up along s,
  φ : blow-up along a sm. p-bisection (say B)
  q : del Pezzo fib'n of degree 9 (=$\mathbb{P}^2$-bundle)



- ∃opposite construction ($-K_Z$ is always $(p \circ \varphi)$-ample)

Sarkisov link

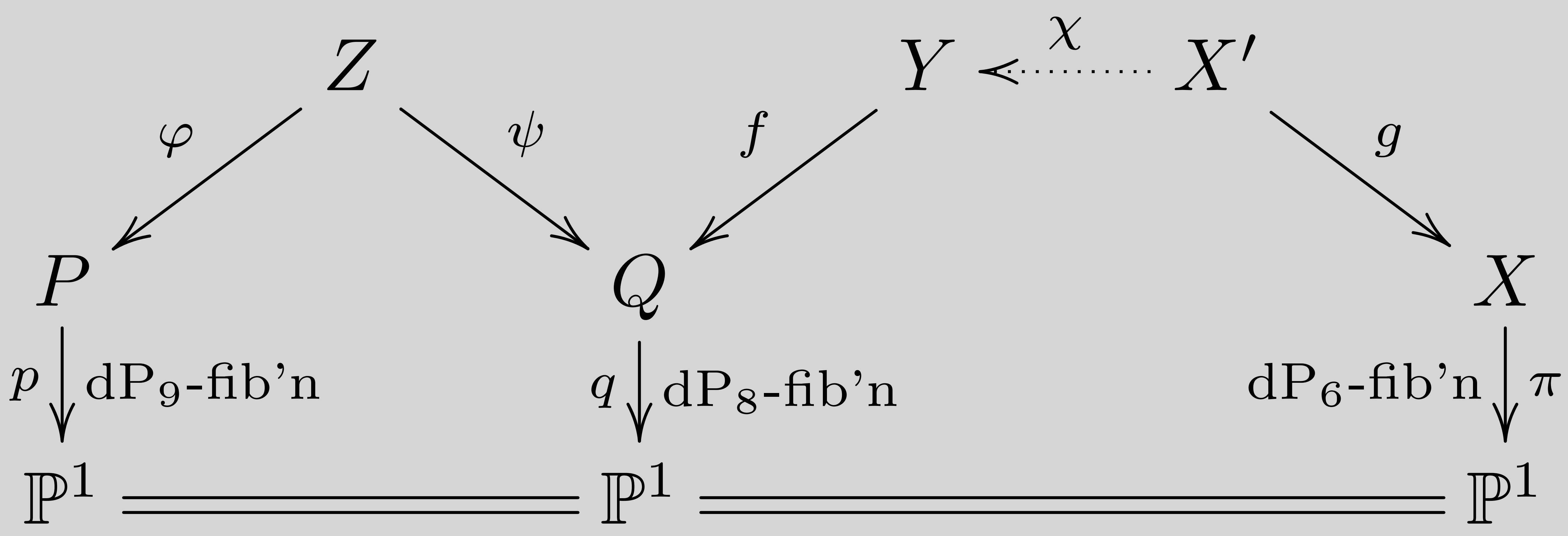
C : smooth plane cubic curve

$\pi\colon X \to \mathbb{P}^1$: del Pezzo fib'n of degree $d = 6$,

B_f : π-fiber, $\exists B_h$: prime divisor on X

s.t • $C \cong$ some π-fiber • $X \backslash (B_f \cup B_h) \cong \mathbb{A}^3$

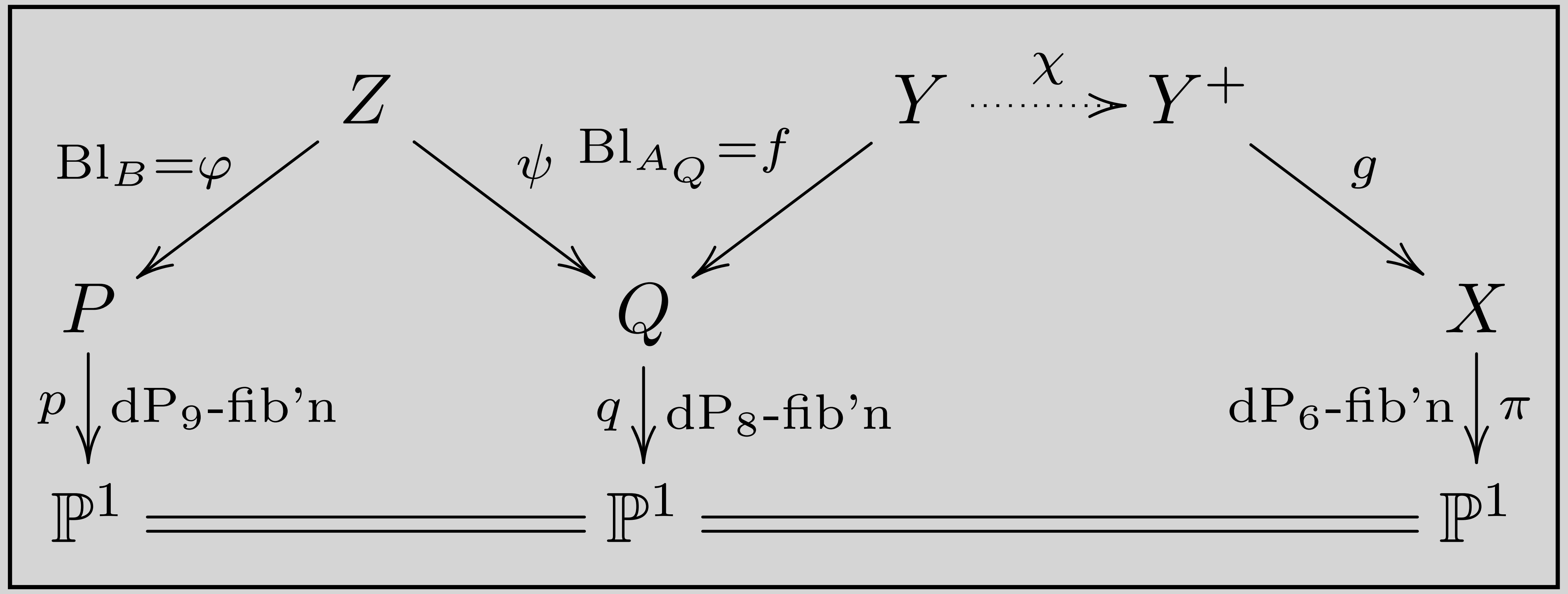
- In addition, assume that $\mathrm{mult}_{s_0} B_h = \mathrm{mult}_s (B_h)_Q = 1$
- Then strict transform of B_h is always linearly equivalent to $-K$ over $\mathbb{P}^1$



- Both B : sm. p-bisection and $A := T_P$: p-trisection are contained in $(B_h)_P$
- $\overline{p}\colon (B_h)_P \to \mathbb{P}^1$ satisfies: $\forall x \in \mathbb{P}^1$, $\overline{p}^*(x)$: smooth $\Rightarrow \overline{p}^*(x) \cong \overline{\pi}^*(x)$

Observation

(★)



- Let $p\colon P \to \mathbb{P}^1$: del Pezzo fib'n of degree 9,
  $S \subset P$: eff. divisor $\sim_{\mathbb{P}^1} -K_P$, $A \subset S$: p-trisection, $B \subset S$: sm. p-bisection.

- Suppose ∃diagram (★) and define $B_h := S_X$ and $\bar{\pi}\colon B_h \to \mathbb{P}^1$

- Then $\bar{p}\colon S \to \mathbb{P}^1$ satisfies: $\forall x \in \mathbb{P}^1$, $\bar{p}^*(x)$: smooth $\Rightarrow \bar{p}^*(x) \cong \bar{\pi}^*(x)$

- To prove Theorem B', it suffices to find plenty of $(P, S, A, B, \infty \in \mathbb{P}^1)$ s.t.
  ∃diagram (★) and X is a completion of $\mathbb{A}^3$ w/ boundary $B_h \cup \pi^{-1}(\infty)$

- Remark: (★) does not preserve $X \backslash (B_h \cup \pi^{-1}(\infty))$

§5 Construction

Theorem D [DKN24] (=Main Theorem)

- $p = \mathrm{pr}_1 : \mathbb{P}^1 \times \mathbb{P}^2 \to \mathbb{P}^1$, $([x_0 : x_1], [z_0 : z_1 : z_2])$: coordinates of $\mathbb{P}^1 \times \mathbb{P}^2$.

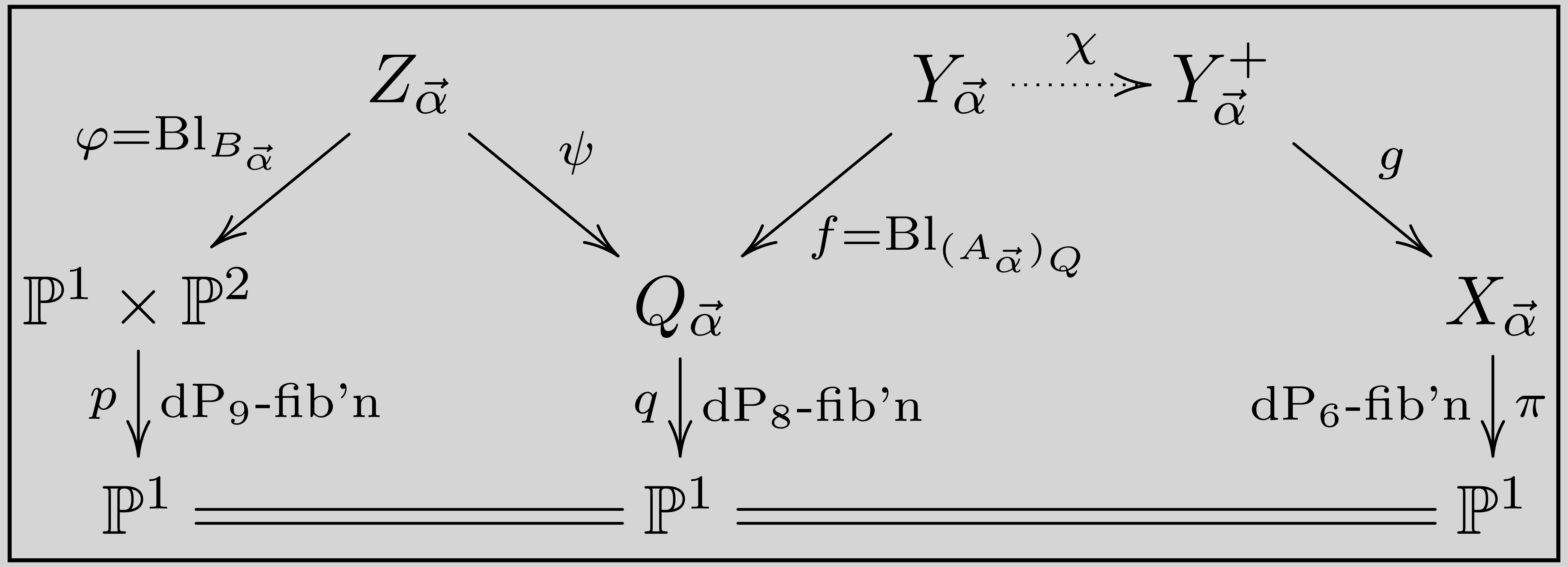
- For $\overrightarrow{\alpha} = (\alpha_3, \alpha_4, \alpha_6, \alpha_7) \in k^4$, take $S_{\vec{\alpha}}$, $A_{\vec{\alpha}}$, $B_{\vec{\alpha}} \subset \mathbb{P}^1 \times \mathbb{P}^2$ as follows.

  $S_{\vec{\alpha}} := \{z_2^3 x_0 + (z_0^3 + z_2(z_1^2 + \alpha_3 z_1 z_2 + \alpha_4 z_0^2 + \alpha_6 z_0 z_1 + \alpha_7 z_0 z_2))x_1 = 0\}$: (1,3)-div.

  $A_{\vec{\alpha}} := \{z_1 = 0\} \cup S_{\vec{\alpha}}$: p-trisection

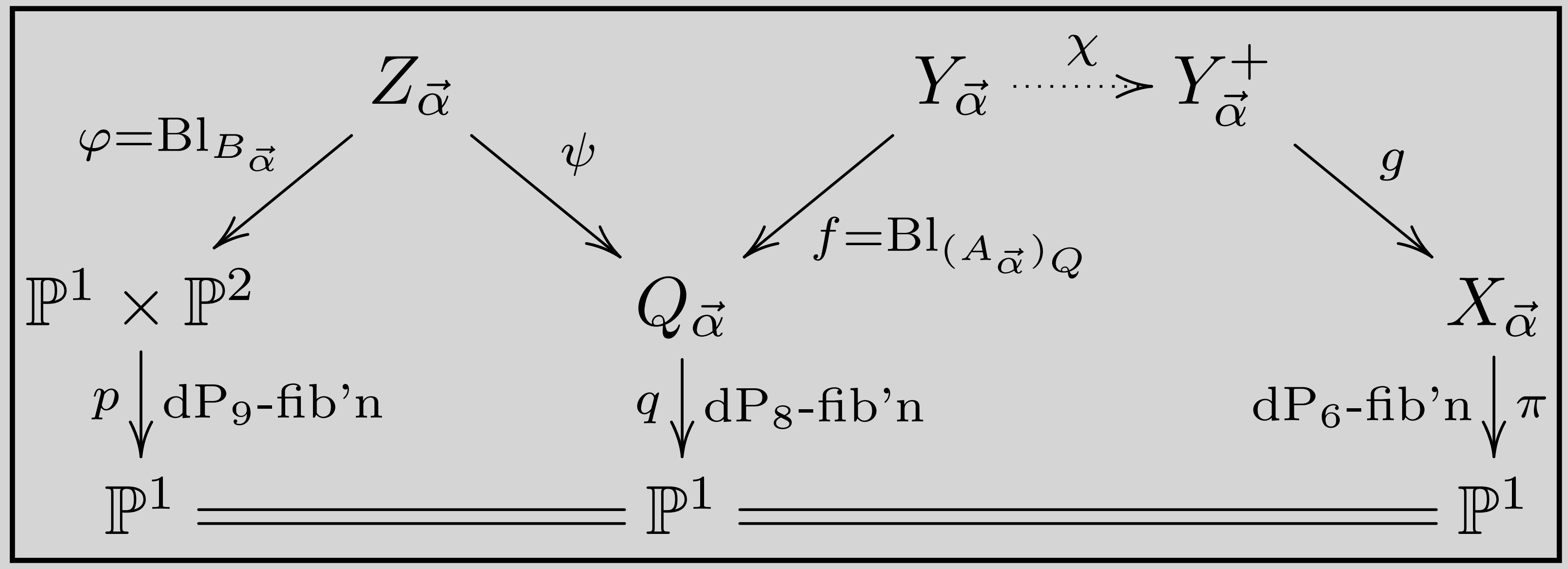
  $B_{\vec{\alpha}} := \{z_0 = 0, z_2^2 x_0 + (z_1^2 + \alpha_3 z_1 z_2)x_1 = 0\} \subset \{z_0 = 0\} \cup S_{\vec{\alpha}}$: p-bisection

- Then (1) ∃diagram (⋆)



Theorem D [DKN24]

(★)



- $p = \mathrm{pr}_1 : \mathbb{P}^1 \times \mathbb{P}^2 \to \mathbb{P}^1$, $([x_0 : x_1], [z_0 : z_1 : z_2])$: coordinates of $\mathbb{P}^1 \times \mathbb{P}^2$.

- For $\overrightarrow{\alpha} = (\alpha_3, \alpha_4, \alpha_6, \alpha_7) \in k^4$, take $S_{\vec{\alpha}}$, $A_{\vec{\alpha}}$, $B_{\vec{\alpha}} \subset \mathbb{P}^1 \times \mathbb{P}^2$ as follows.

  $S_{\vec{\alpha}} := \{z_2^3 x_0 + (z_0^3 + z_2(z_1^2 + \alpha_3 z_1 z_2 + \alpha_4 z_0^2 + \alpha_6 z_0 z_1 + \alpha_7 z_0 z_2))x_1 = 0\}$: (1,3)-div.

  $A_{\vec{\alpha}} := \{z_1 = 0\} \cup S_{\vec{\alpha}}$: p-trisection

  $B_{\vec{\alpha}} := \{z_0 = 0, z_2^2 x_0 + (z_1^2 + \alpha_3 z_1 z_2)x_1 = 0\} \subset \{z_0 = 0\} \cup S_{\vec{\alpha}}$: p-bisection

- Then (1) ∃diagram (★)

  (2) X is a completion of $\mathbb{A}^3$ w/ boundary $B_{h,\vec{\alpha}} \cup B_{f,\vec{\alpha}} := (S_{\vec{\alpha}})_{X_{\vec{\alpha}}} \cup \pi^{-1}([1:0])$

Theorem D ⇒ Theorem B'

- $\overline{p} : S \to \mathbb{P}^1$: induced morphism

- $\overrightarrow{\alpha} = (0,0,0,0) \Rightarrow S_{\vec{\alpha}} := \{z_2^3 x_0 + (z_0^3 + z_2 z_1^2) x_1 = 0\}$

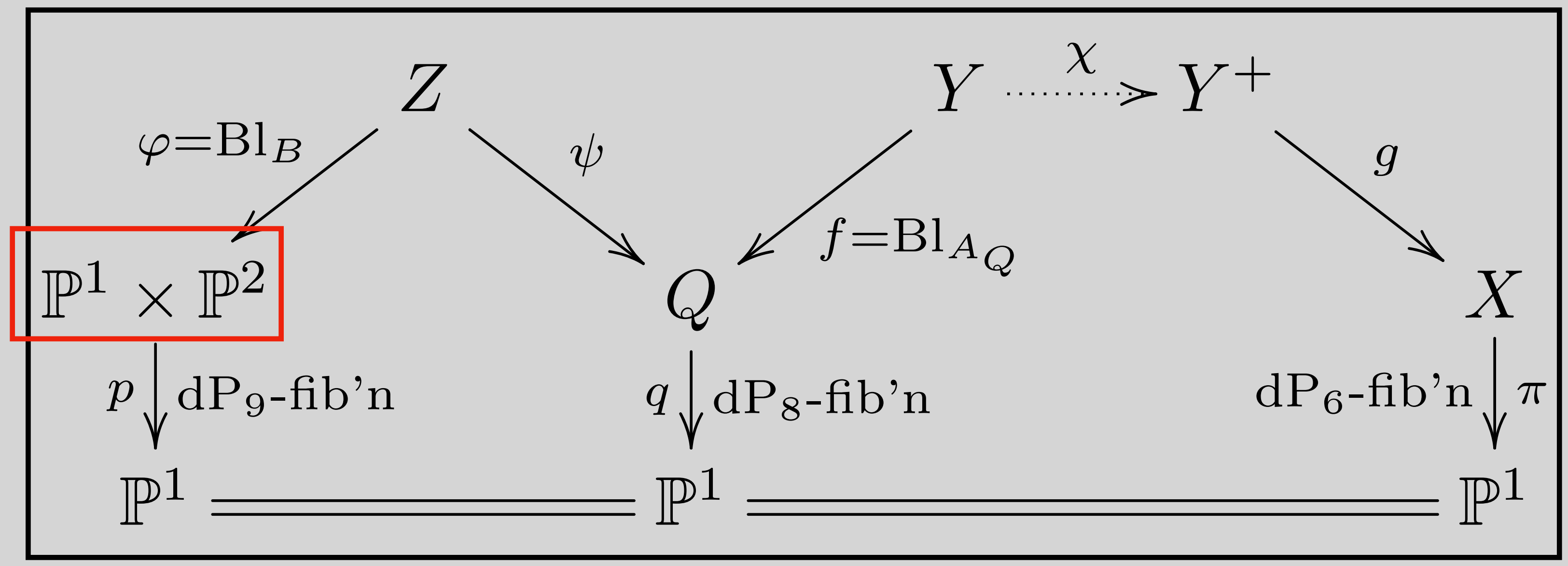
  $\Rightarrow \overline{p}^*([a:1])$ is a smooth ell. curve w/ $j = 0$ $(\forall a \in k)$

- $\overrightarrow{\alpha} = (0,0,0,1) \Rightarrow S_{\vec{\alpha}} := \{z_2^3 x_0 + (z_0^3 + z_2(z_1^2 + z_0 z_2)) x_1 = 0\}$

  $\Rightarrow \overline{p}^*([a:1])$ is a smooth ell. curve w/ $j = \dfrac{1728 \cdot 4}{4 + 27a^2}$ $(\forall x \in k \backslash \{\pm(2\sqrt{-3})/9\})$

- $\forall x \in \mathbb{P}^1, \overline{p}^*(x)$: smooth $\Rightarrow \overline{p}^*(x) \cong \overline{\pi}^*(x)$ $\square$

Proof of Theorem D

(★) 

- We omit subscript $\vec{\alpha}$

- Properties:

  $\mathrm{Sing}\ S = \{([1:0],[0:1:0])\}$, $A \cong B \cong \mathbb{P}^1$,

  $S \cap \{z_2 = 0\} = \{z_0 = z_2 = 0\}$: p-section,

  $S \cap \{z_0 = 0\} = \{z_0 = z_2 = 0\} \cup B$,

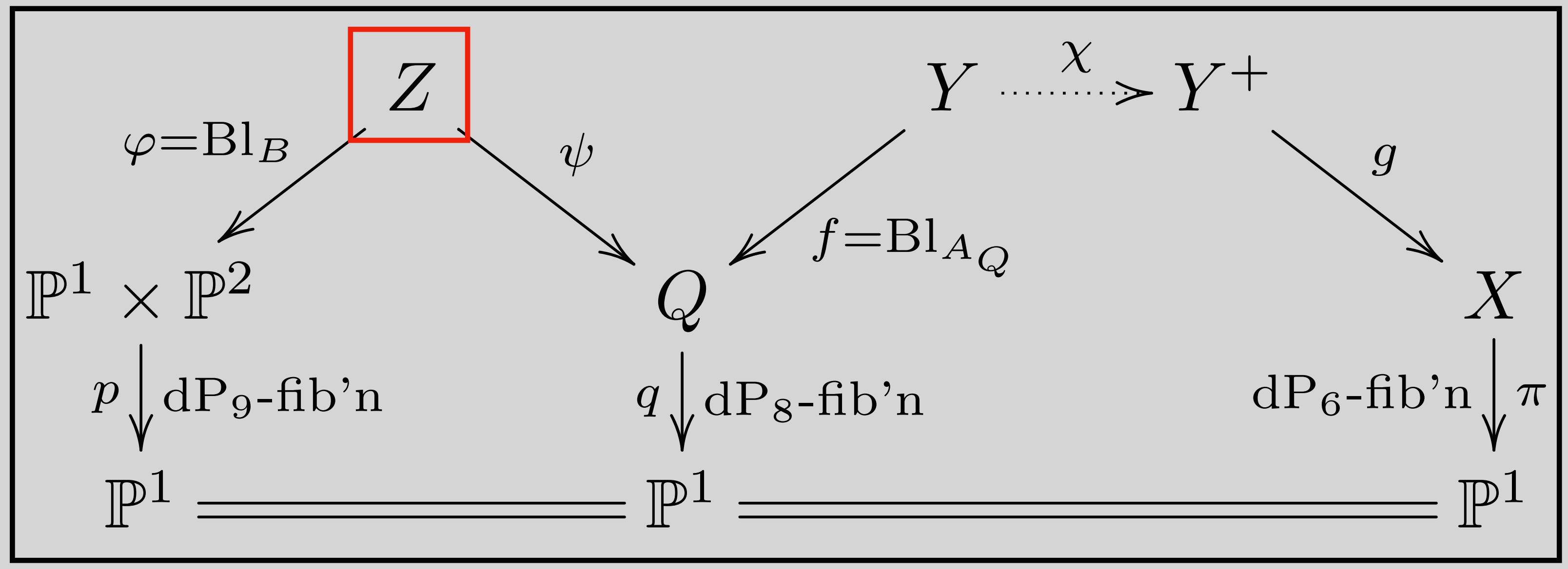
  $P \backslash (\{z_2 = 0\} \cup \{x_1 = 0\}) \cong \mathbb{A}^3$

  $\supset S \backslash (\{z_2 = 0\} \cup \{x_1 = 0\}) \cong \mathbb{A}^2$: linearizable

  $\supset A \backslash (\{z_2 = 0\} \cup \{x_1 = 0\}) \cong \mathbb{A}^1$, $B \backslash (\{z_2 = 0\} \cup \{x_1 = 0\}) \cong \mathbb{A}^1$.

Proof of Theorem D

(★)



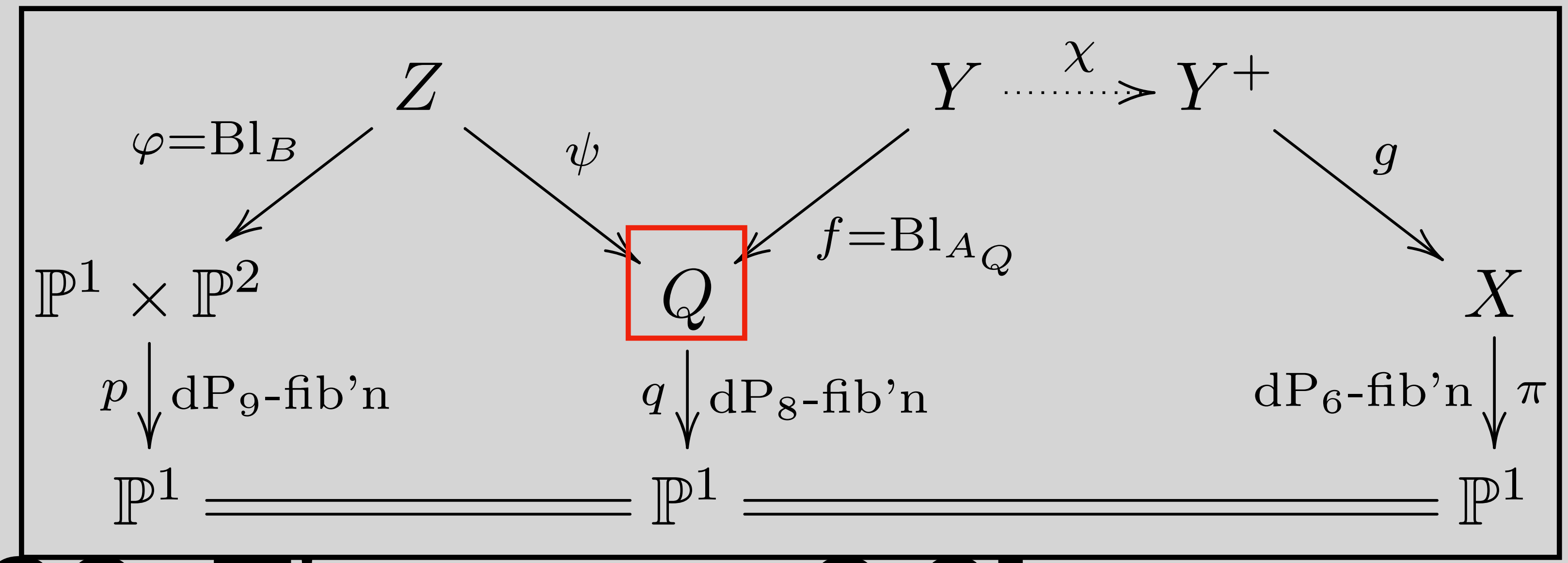
- $\varphi : Z \to \mathbb{P}^1 \times \mathbb{P}^2$: blow-up along B
  (Fano 3-fold of No.21 in [Mori-Mukai81, Table 3])

- E_φ : φ-excep. divisor

- $|\varphi^* \mathcal{O}_{\mathbb{P}^1 \times \mathbb{P}^2}(2,2) - E_\varphi|$ defines $\psi : Z \to Q$: birational extremal contraction

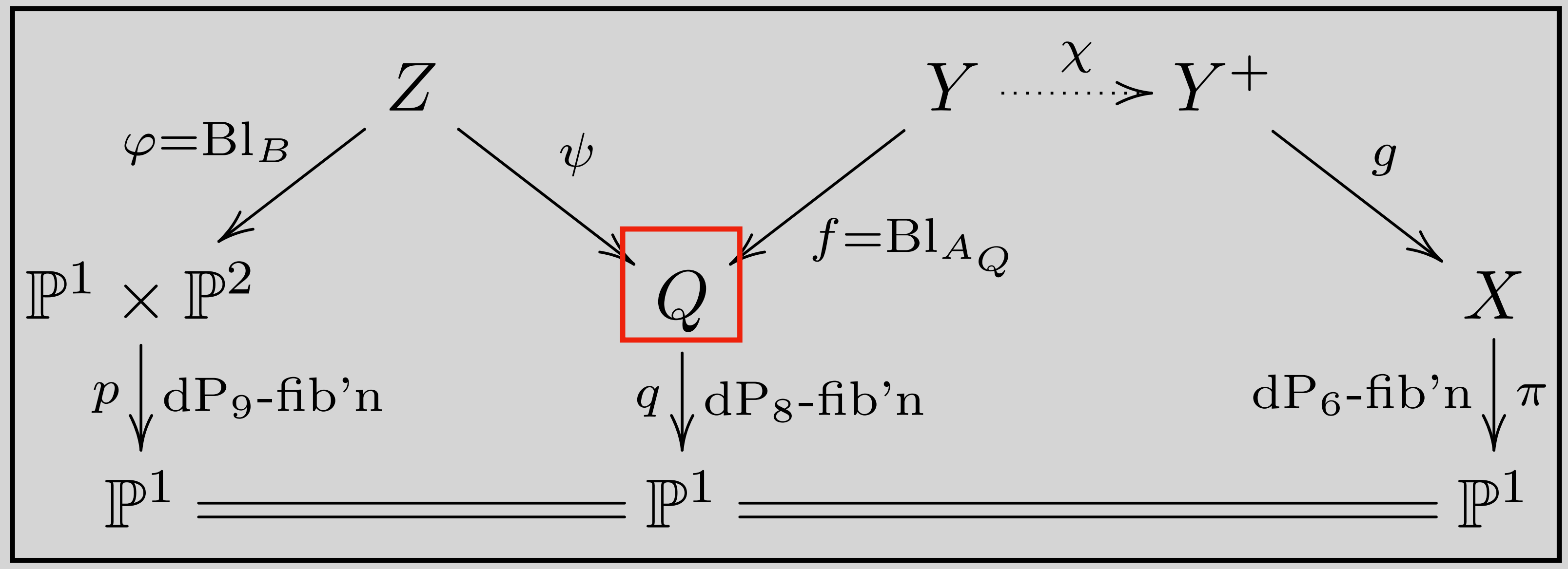
- E_ψ = strict transform of $\{z_0 = 0\}$ in Z.

Proof of Theorem D

(★) 

- Q: weak Fano 3-fold of type (2.3.2) in [Takeuchi22, Theorem 2.3] endowed with a dP_8-fib'n $q\colon Q \to \mathbb{P}^1$
- $s = \psi(E_\psi)$: q-section & the unique $(-K_Q)$-trivial curve
- $\exists \mathcal{O}_Q(1)$: Cartier div. s.t $-K_Q \sim \mathcal{O}_Q(2)$
- Q is embedded into $\mathbb{F}(1,1,1,0) := \mathbb{P}_{\mathbb{P}^1}(\mathcal{O}_{\mathbb{P}^1}(1)^{\oplus 3} \oplus \mathcal{O}_{\mathbb{P}^1})$ by $|\mathcal{O}_Q(1)|$
- $\mathbb{P}^1 \times \mathbb{P}^2 \dashrightarrow \mathbb{F}(1,1,1,0)$ is defined by $|\mathcal{O}_{\mathbb{P}^1\times\mathbb{P}^2}(1,2) \otimes \mathcal{I}_B|$

Proof of Theorem D

(★) 

- Let $S_f := q^{-1}([1:0])$

- S_h, (resp. H_1, H_2, T): the strict transform of S, (resp.$\{z_1 = 0\}$, $\{z_2 = 0\}$, A) in Q

- $\exists$coordinates $([x_0 : x_1], [w_0 : w_1 : w_2 : w_3])$ of $\mathbb{F}(1,1,1,0)$ s.t.

$Q = \{w_2^2 x_0 + x_1(w_1^2 + \alpha_3 w_1 w_2) - w_0 w_3 = 0\}$

$S_h = Q \cap \{x_1(w_0^2 + w_2(\alpha_4 w_0 + \alpha_6 w_1 + \alpha_7 w_2)) + w_2 w_3 = 0\}$ $\sim \mathcal{O}_Q(2) - S_f$

$H_i = Q \cap \{w_i = 0\}$ (for $i = 1,2$) $\sim \mathcal{O}_Q(1) - S_f$

$T = S_h \cap H_1 \cap \{w_3^2 + x_1(w_0 w_2 x_0 + \alpha_4 w_0 w_3 + \alpha_7 w_2 w_3) = 0\}$: sm. q-trisection

$s = \{w_0 = w_1 = w_2 = 0\}$: q-section

Proof of Theorem D

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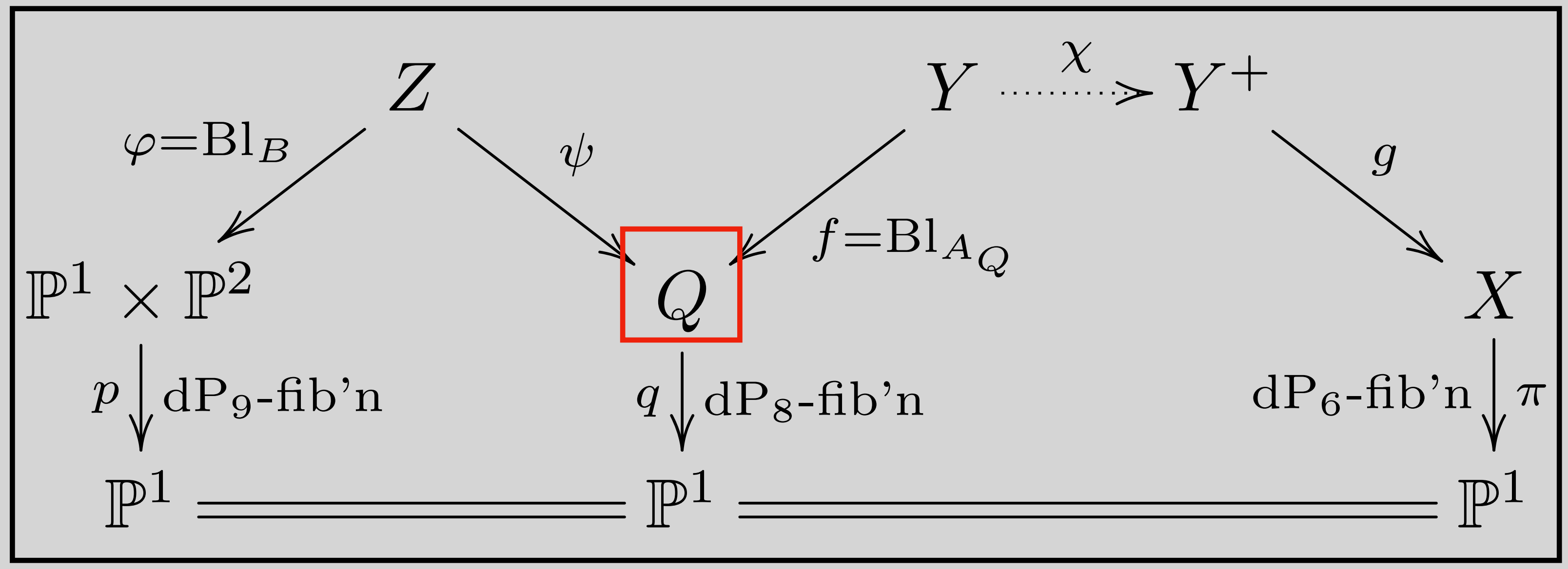


- Properties: $S_h \cap H_1 = s \sqcup T$ (schematically)

  $S_h \cap H_2 = 4s +$ (curves in S_f)

- $Q^\circ := Q \backslash (H_2 \cup S_f)$ ($\cong \{x' + (w_1')^2 + \alpha_3 w_1' - w_0' w_3' = 0\}$ in $\mathbb{A}^4_{[x', w_0', w_1', w_3']}$) $\cong \mathbb{A}^3_{[w_0', w_1', w_3']}$

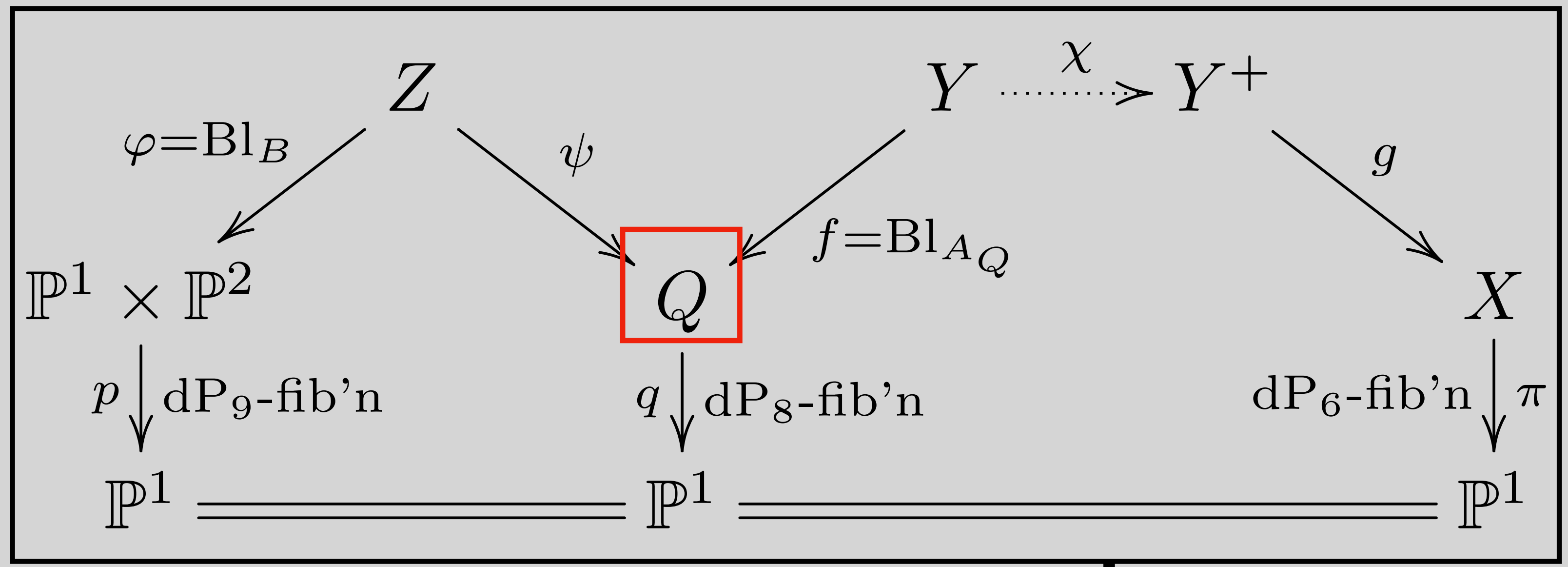
  $\{v_1 = 0\} =$ $H_1 \cap Q^\circ \cong \{w_1' = 0\} \cong \mathbb{A}^2$,

  $\{v_2 = 0\} =$ $S_h \cap Q^\circ \cong \{(w_0')^2 + \alpha_4 w_0' + \alpha_6 w_1' + \alpha_7 + w_3' = 0\} \cong \mathbb{A}^2$

  $\{v_1 = v_2 = 0\} =$ $(H_1 \cap S_h) \cap Q^\circ = T \cap Q^\circ$ ($\because s \subset H_2$)

- **Key lemma : $\exists$coordinates $[v_1, v_2, v_3]$ of $Q^\circ \cong \mathbb{A}^3$ s.t.**

Proof of Theorem D

(★) 

- H_i : smooth $(i = 1,2)$ and $H_1 \cap H_2 = s \sqcup (s' := \{w_1 = w_2 = w_3 = 0\})$: two q-section

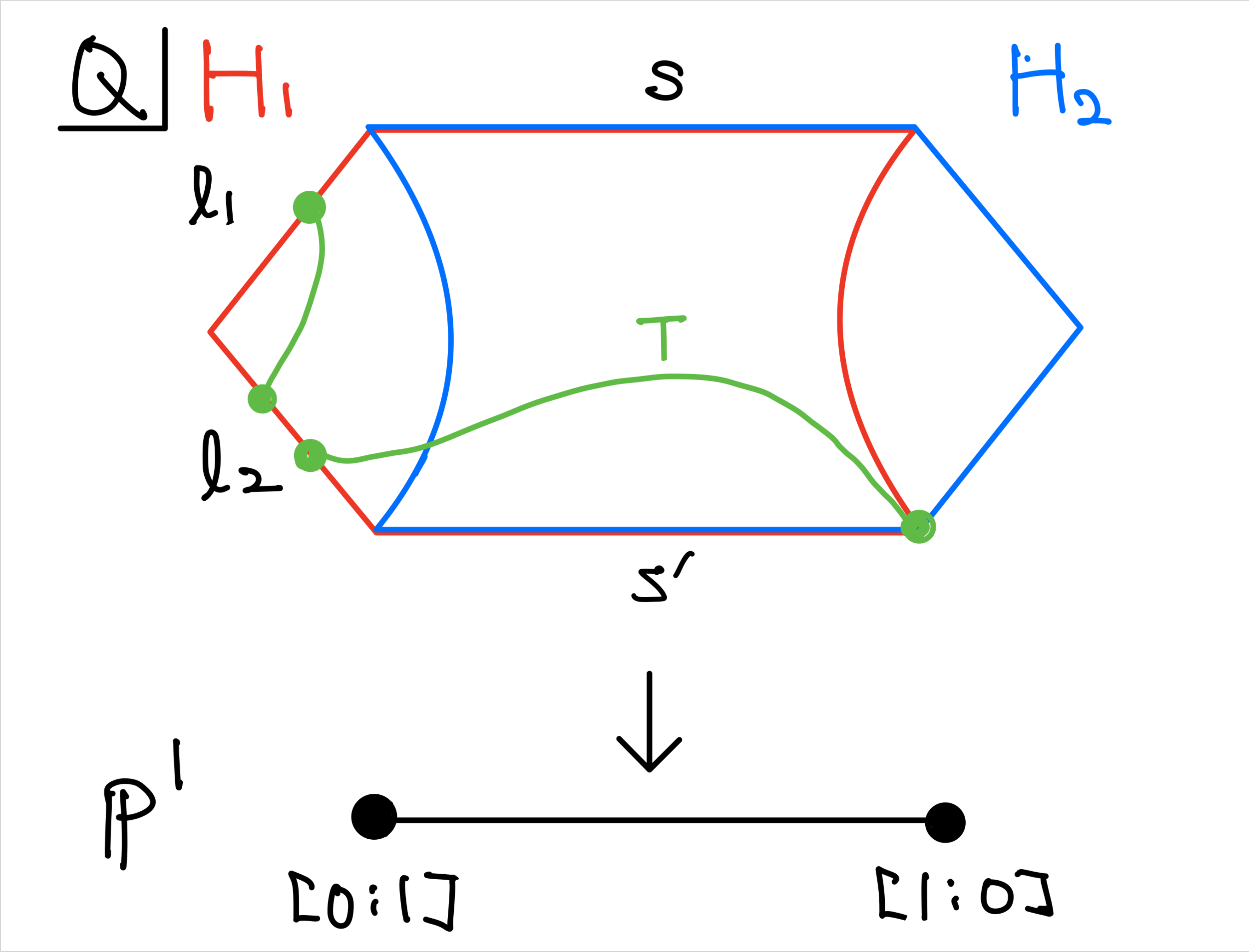
- conic bundle $q|_{H_1}$ (resp. $q|_{H_2}$) has the unique singular fiber, which is reducible and over $[0:1]$ (resp. $[1:0]$)

- $(q|_{H_1})^{-1}([0:1]) = \{x_0 = w_0 = w_1 = 0\} =: l_1$

  $\cup \{x_0 = w_1 = w_3 = 0\} =: l_2$

- $(l_i \cdot T)_{H_1} = i$ for $i = 1,2$

- $T \cap H_2 = \{([1:0],[1:0:0:0])\} \subset S_f$



Proof of Theorem D

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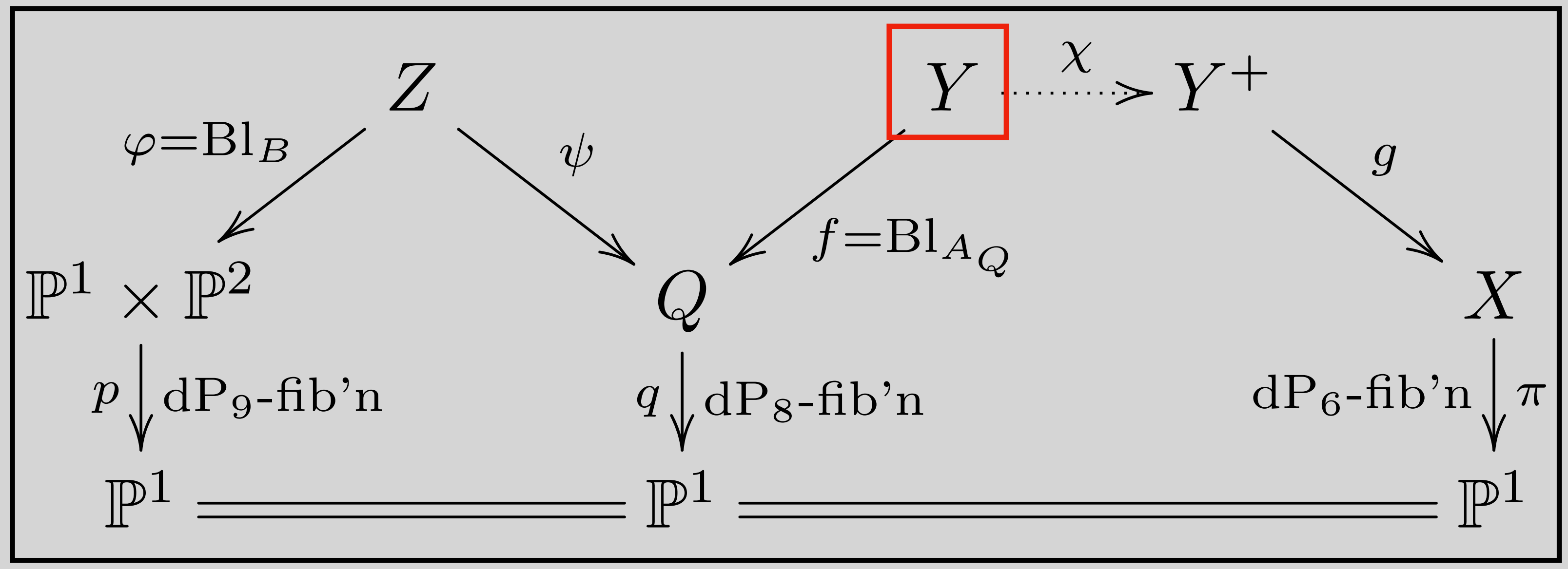

- $f\colon Y \to Q$: blow-up along T

- Property: $-K_Y$ is nef and big (i.e., Y is weak Fano)

$\because\ -K_Y \sim f^*S_f + (S_h)_Y \quad \Rightarrow \forall(-K_Y)$-negative curve $\subset (S_h)_Y$

$\sim f^*H_1 + 2f^*S_f + (H_1)_Y \quad \Rightarrow \forall(-K_Y)$-negative curve $\subset s_Y \cup (H_1)_Y = (H_1)_Y$

$(S_h)_Y \cap (H_1)_Y = s_Y$ and $(-K_Y \cdot s_Y) = (-K_Q \cdot s) = 0 \quad \Rightarrow -K_Y$: nef

$(-K_Y)^3 = 22 \Rightarrow -K_Y$: big

Proof of Theorem D

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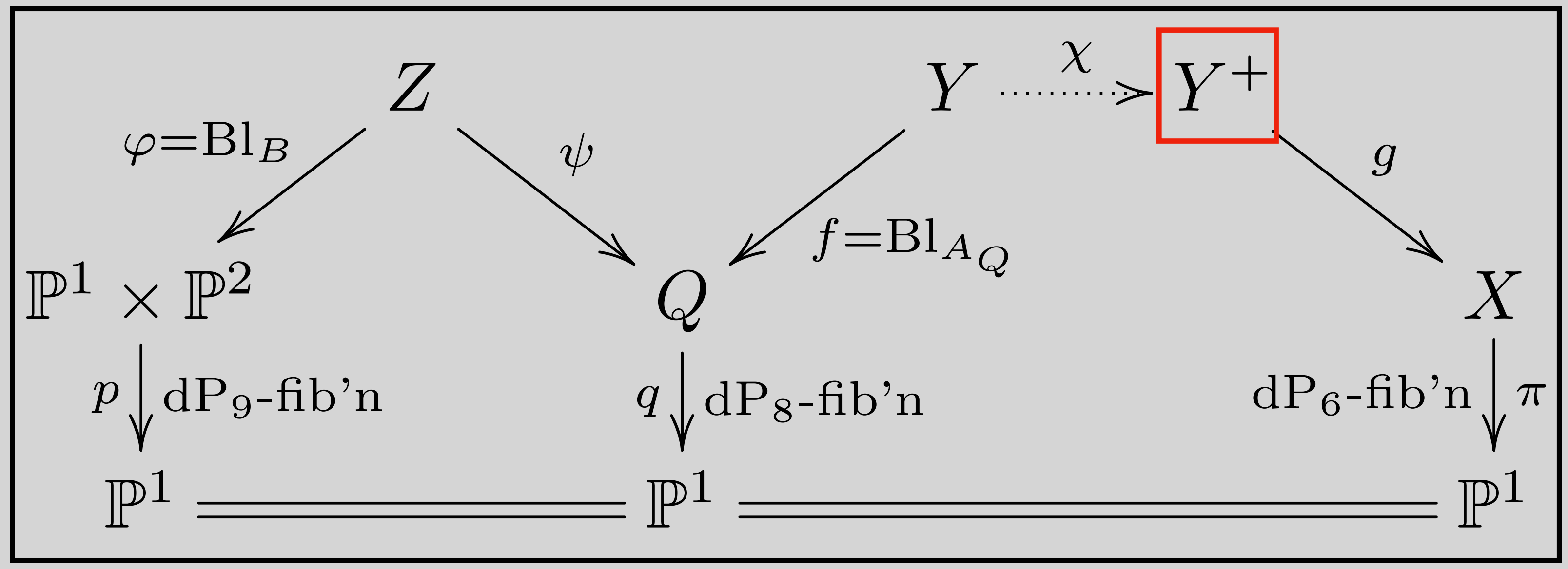
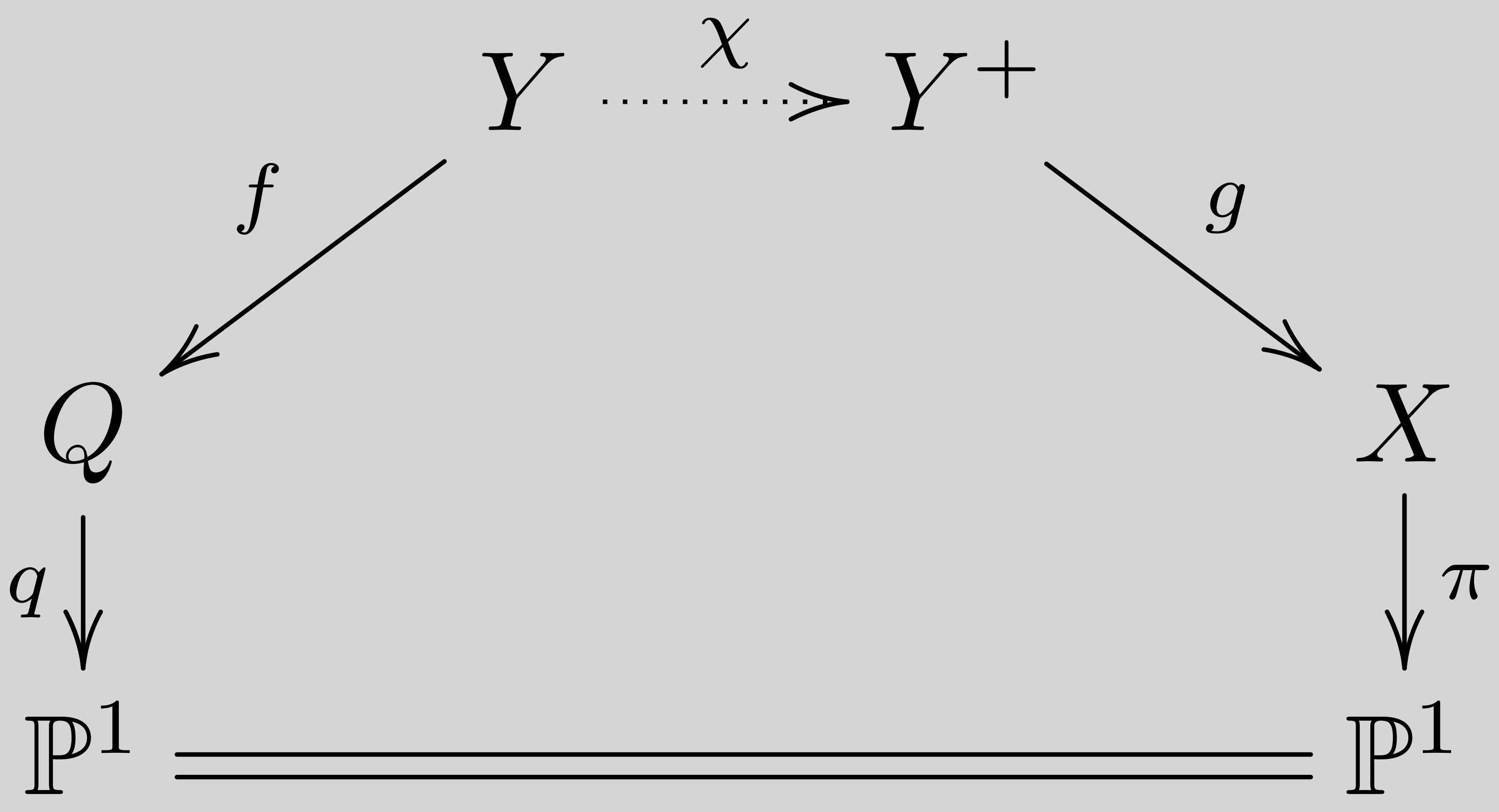


- By [Fukuoka17], ∃diagram

  w/ χ : the flop/$\mathbb{P}^1$, π : dP_6-fib'n,

  g : blow-up along a section, say s_0

  $E_g = (H_1)_{Y^+}$

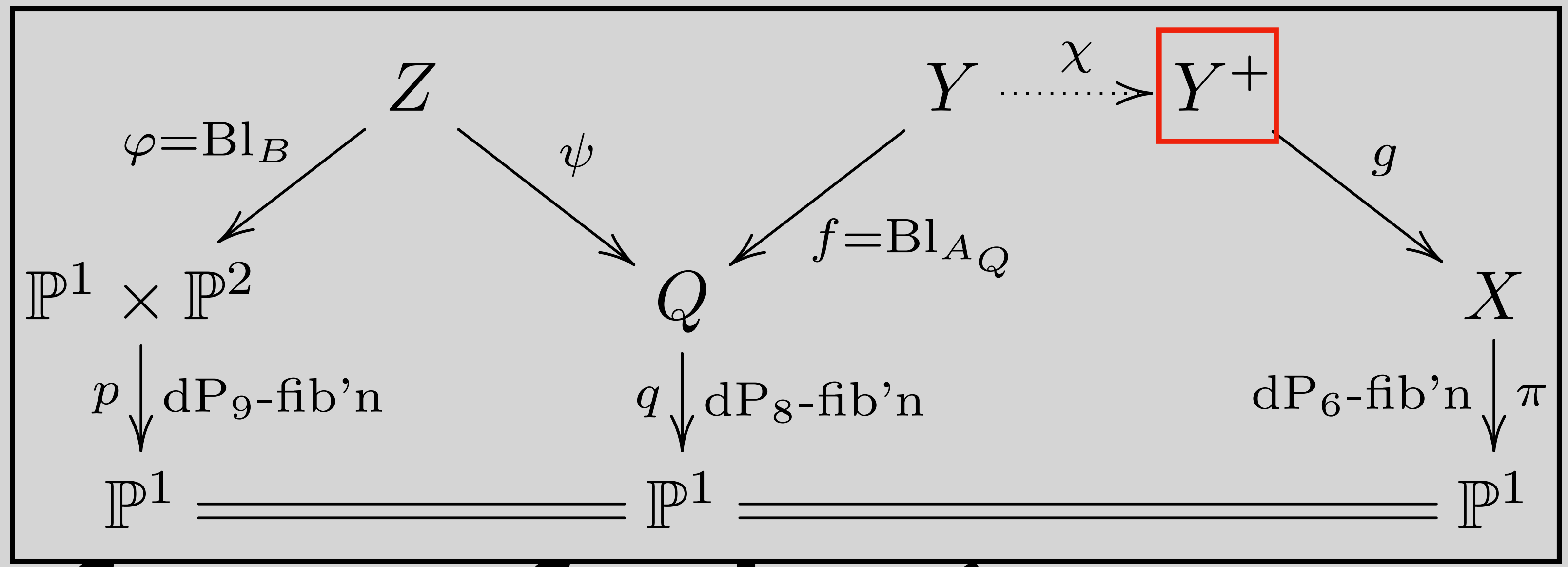


- $B_h := (S_h)_X$, $B_f := (S_f)_X$

- $X \backslash (B_h \cup B_f) \cong Y^+ \backslash ((S_h)_{Y^+} \cup (S_f)_{Y^+} \cup (H_1)_{Y^+})$

  "=" $Y^+ \backslash ((S_h)_{Y^+} \cup (S_f)_{Y^+} \cup (H_1)_{Y^+} \cup (H_2)_{Y^+}) + (H_2)_{Y^+} \backslash ((S_h)_{Y^+} \cup (S_f)_{Y^+} \cup (H_1)_{Y^+})$

Proof of Theorem D

(★) 

- **χ is the the Atiyah flop of $l :=$ (the strict transform of l_2 in Y)**

  $\because$ Suppose $r \subset Y$ is a flopping curve ($\Rightarrow r$ is not f-exceptional)

  $$-K_Y \sim f^*H_1 + 2f^*S_f + (H_1)_Y \Rightarrow 0 = (-K_Y \cdot r)_Y = (H_1, r_Q)_Q + ((H_1)_Y \cdot r)_Y$$

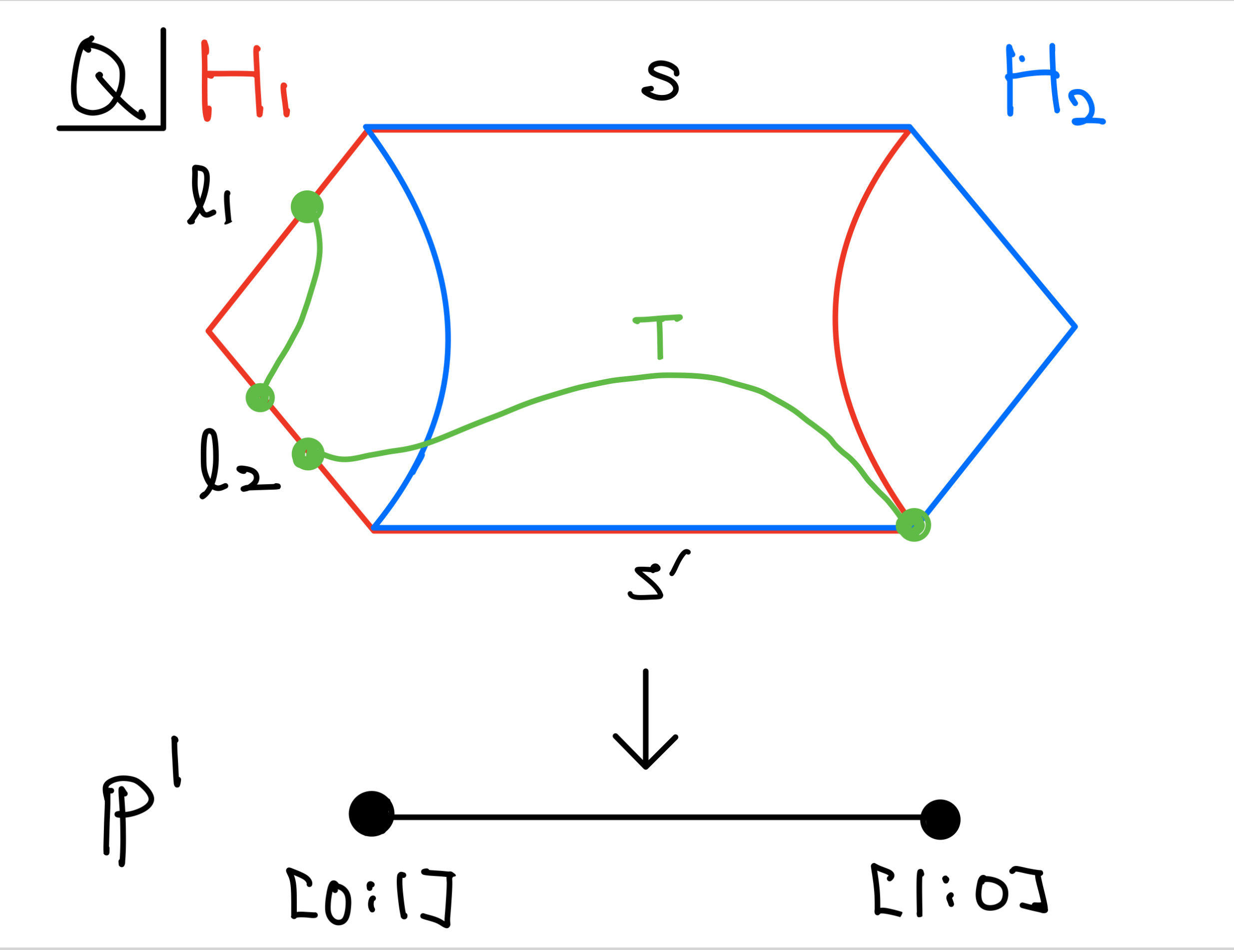
  $$> ((H_1)_Y \cdot r) \qquad \therefore r \subset (H_1)_Y$$

  If r_Q is a smooth $q|_{H_1}$-fiber, then $(-K_Y \cdot r)_Y = (-K_Q, r_Q)_Q - (T \cdot r)_{H_1} = 1$ ↯

  If $r_Q = l_i$, then $(-K_Y \cdot r)_Y = (-K_Q, r_Q)_Q - (T \cdot r)_{H_1} = 2 - i \qquad \therefore r_Q = l_2$

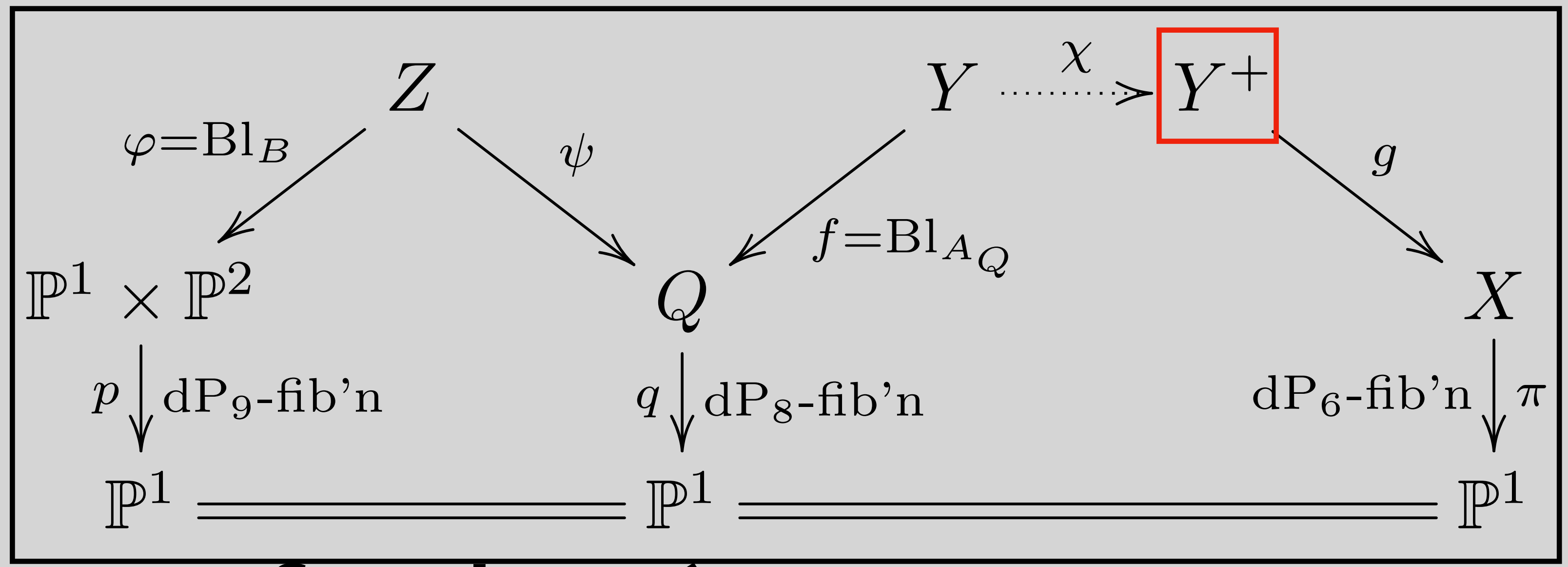
  $$0 \longrightarrow N_l(H_1)_Y \longrightarrow N_lY \longrightarrow (N_{(H_1)_Y}Y)|_l \longrightarrow 0 \quad \therefore N_lY \cong \mathcal{O}_l(-1)^{\oplus 2}$$

  where $N_l(H_1)_Y = \mathcal{O}_l(-1)$ and $(N_{(H_1)_Y}Y)|_l = \mathcal{O}_l(-1)$.



Proof of Theorem D

$(\star)$



- χ is the the Atiyah flop of $l :=$ (the strict transform of l_2 in Y)

- $l \subset (H_1)_Y$ and $l^+ \subset (H_2)_{Y^+}$ ($\because$ $l_2 \cap H_2 \neq \emptyset$ and $T \cap H_2 \subset S_f \Rightarrow l \cap (H_2)_Y \neq \emptyset$)

- $Y^+ \backslash ((S_h)_{Y^+} \cup (S_f)_{Y^+} \cup (H_1)_{Y^+} \cup (H_2)_{Y^+}) \cong Y \backslash ((S_h)_Y \cup (S_f)_Y \cup (H_1)_Y \cup (H_2)_Y)$

  = (Blow-up of $Q^\circ = Q \backslash (H_2 \cup S_f)$ along $T \cap Q^\circ$)

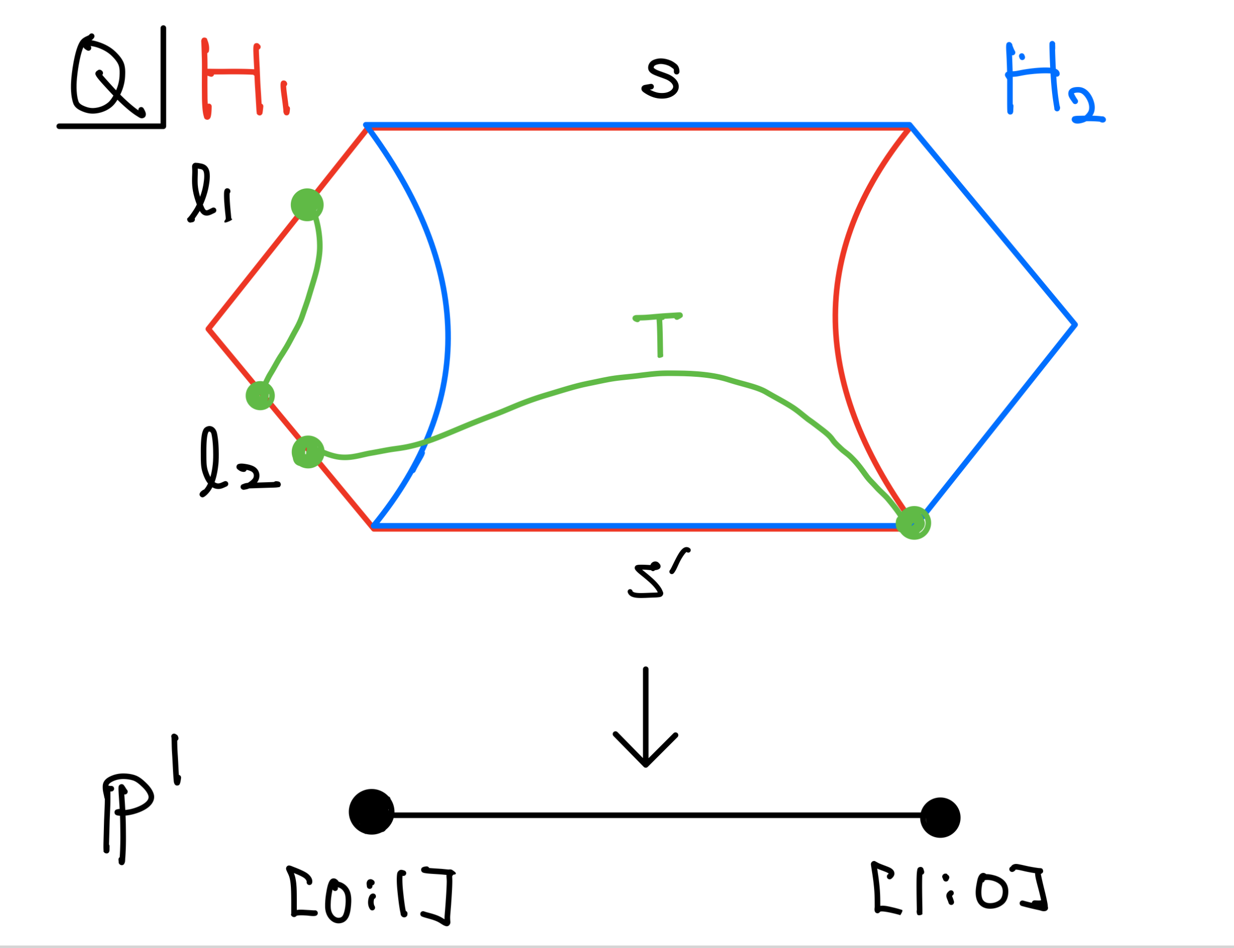
  \(strict transform of $(H_1 \cap Q^\circ) \cup (S_h \cap Q^\circ)$)

  $\cong \mathbb{A}^2 \times (\mathbb{A}^1)^*$

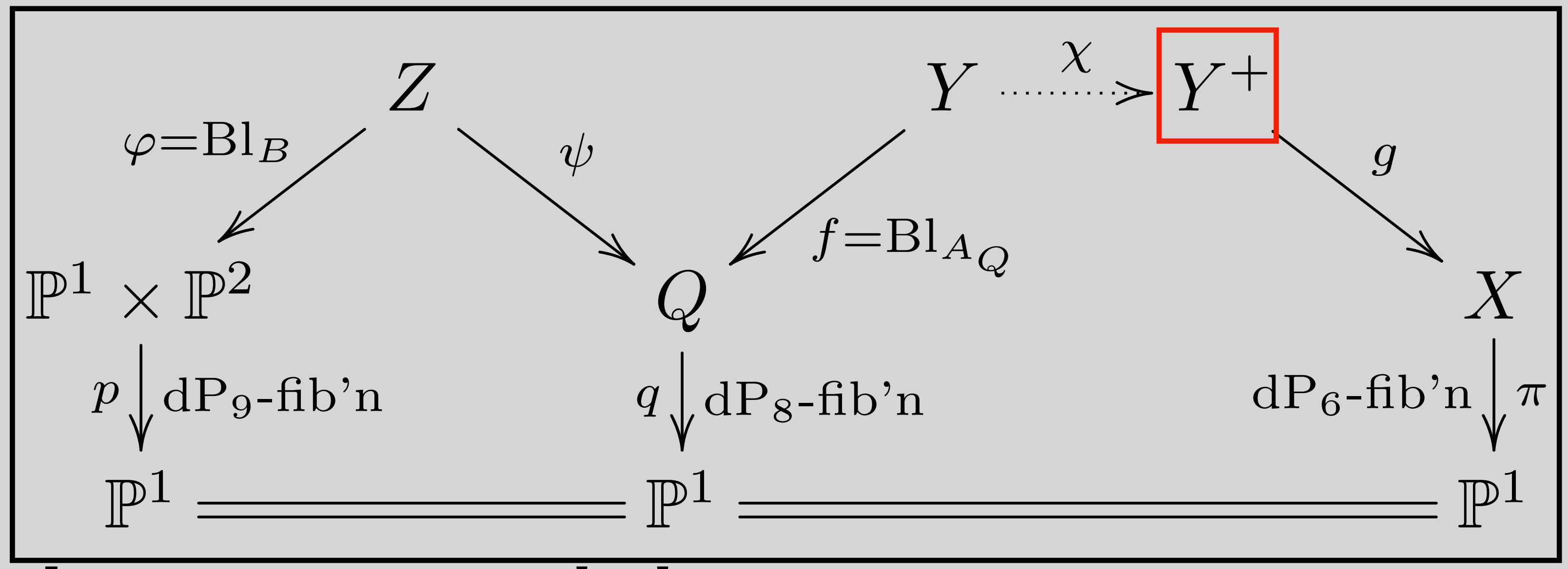
- Key Lemma:

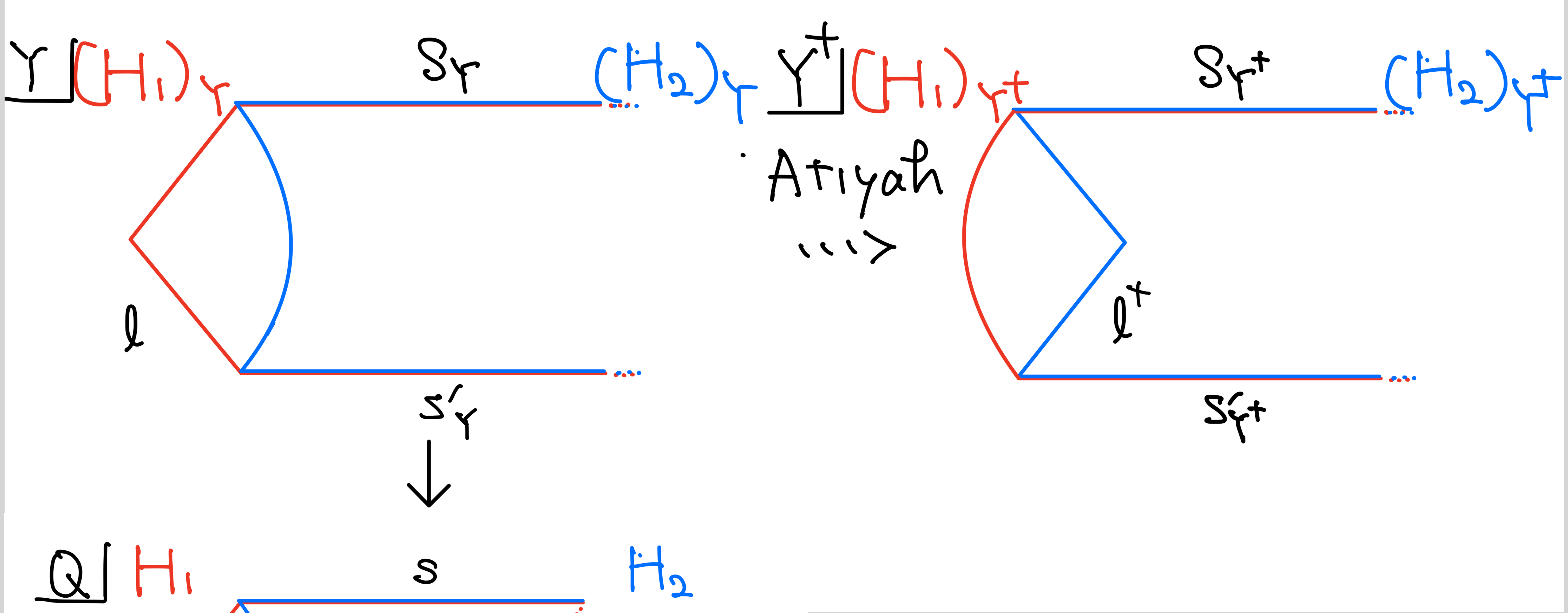
  $\exists$coordinates $[v_1, v_2, v_3]$ of $Q \backslash (H_2 \cup S_f)$

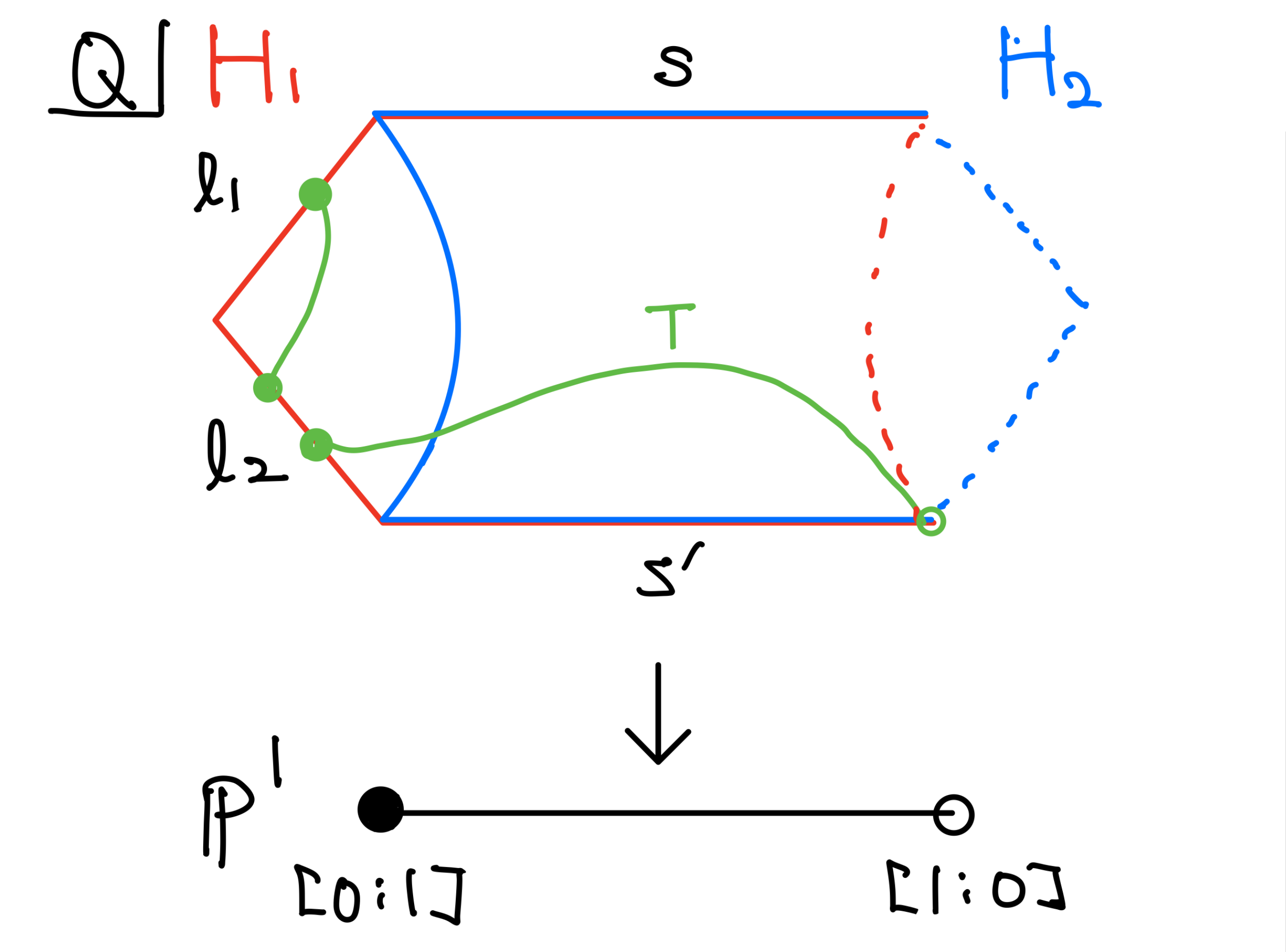
  $\{v_1 = 0\} = H_1 \cap Q^\circ$, $\{v_2 = 0\} = S_h \cap Q^0$, $\{$



Proof of Theorem D

(★) 





- $(H_2)_{Y^+} \backslash (S_f)_{Y^+}$ is isom. to blow-up of $\mathbb{A}^1 \times \mathbb{P}^1$ at a point ($= l \cap s'_Y$)

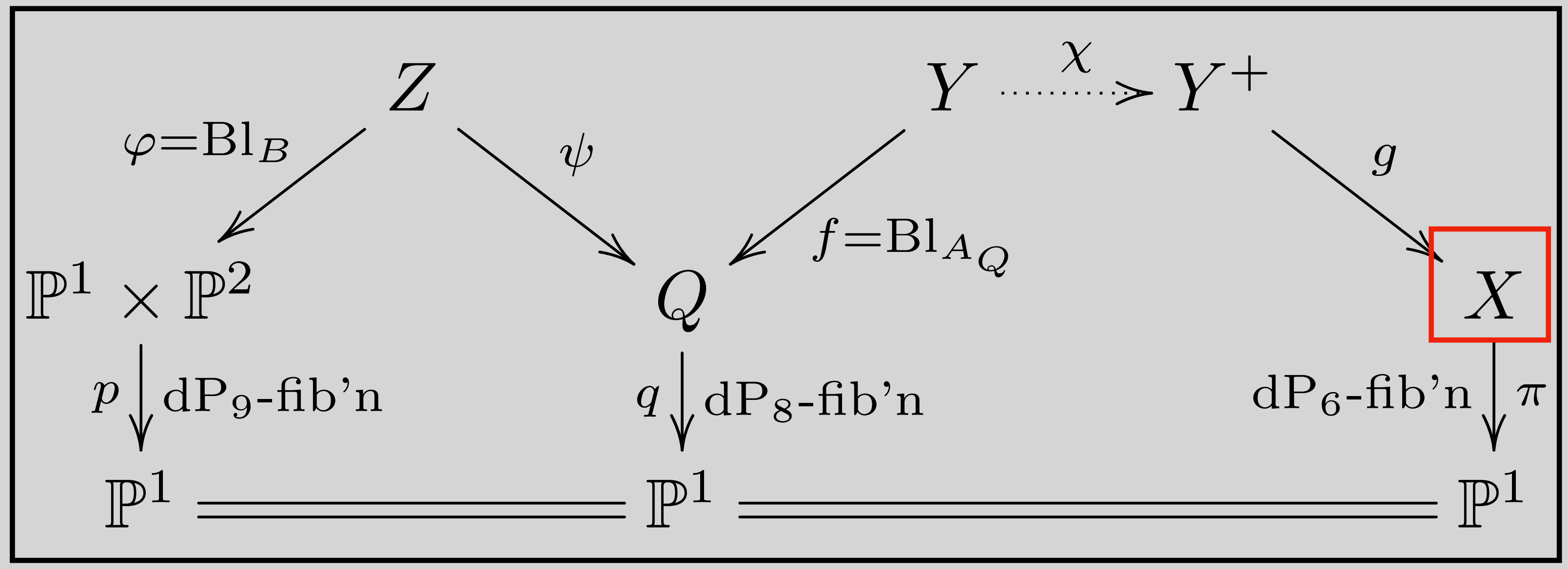
- $S_h \cap H_2 = 4s +$ (curves in S_f)

$\Rightarrow (S_h)_{Y^+} \cap (H_2)_{Y^+} = s_{Y^+} +$ (curves in $(S_f)_{Y^+}$)

- $H_1 \cap H_2 = s \sqcup s' \Rightarrow (H_1)_{Y^+} \cap (H_2)_{Y^+} = s_{Y^+} \sqcup s'_{Y^+}$

- $(H_2)_{Y^+} \backslash ((S_h)_{Y^+} \cup (S_f)_{Y^+} \cup (H_1)_{Y^+}) \cong \mathrm{Bl}_t(\mathbb{A}^1 \times \mathbb{P}^1) \backslash (s_{Y^+} \cup s'_{Y^+}) \cong \mathbb{A}^2$

Proof of Theorem D

(★) 

- $X \backslash (B_h \cup B_f)$

  "=" $Y^+ \backslash ((S_h)_{Y^+} \cup (S_f)_{Y^+} \cup (H_1)_{Y^+} \cup (H_2)_{Y^+})$ + $(H_2)_{Y^+} \backslash ((S_h)_{Y^+} \cup (S_f)_{Y^+} \cup (H_1)_{Y^+})$

  "=" $\mathbb{A}^2 \times (\mathbb{A}^1)^* + \mathbb{A}^2 \cdots$ ①

- $-K_Q \sim S_h + S_f$ & $T \subset S_h \Rightarrow -K_X \sim B_h + B_f$: π-ample $\Rightarrow X \backslash (B_h \cup B_f) = \mathrm{Spec}(\exists \mathfrak{A})$

- $\{B_h, B_f\}$: $\mathbb{Z}$-basis of $\mathrm{Pic}(X) \Rightarrow \mathfrak{A}$: UFD, & $\mathfrak{A}^* = k^* \cdots$ ②

Proof of Theorem D

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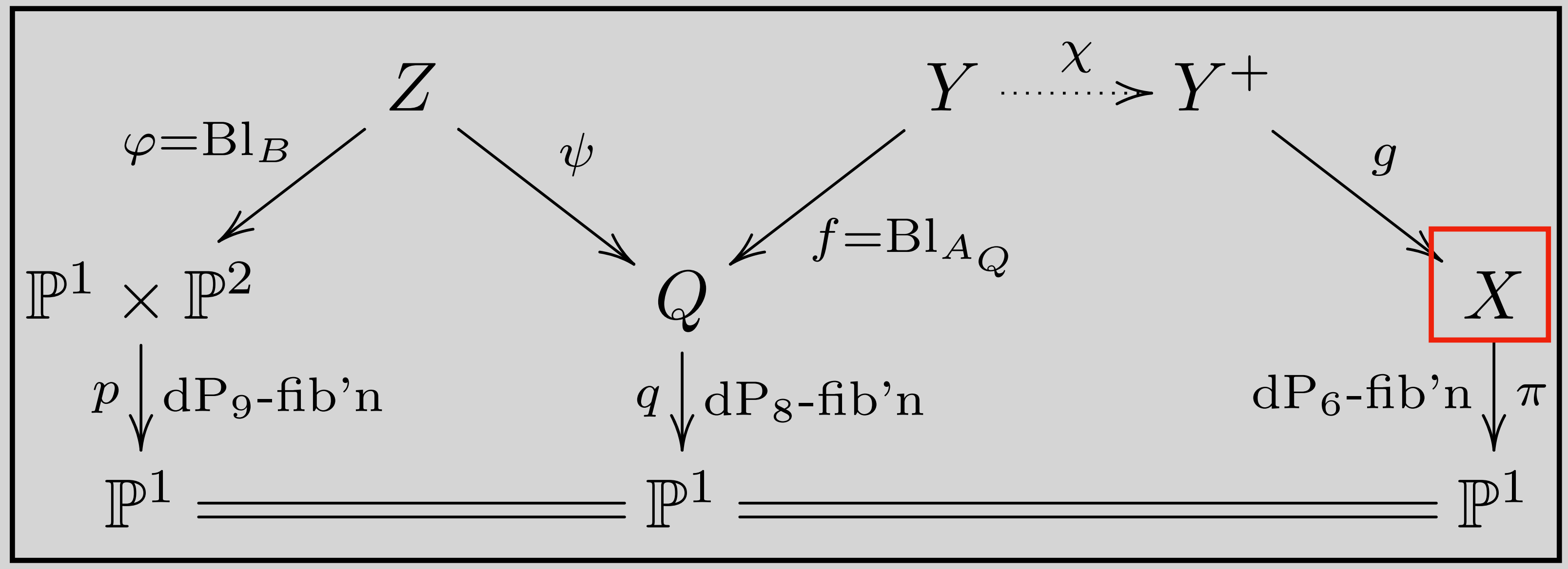


- ① +② $\Rightarrow \exists t \in \mathfrak{A}$ s.t. $\mathfrak{A}[t^{-1}] \cong k[z, z^{-1}][x, y]$ & $\mathfrak{A}/(t) \cong k[x', y']$ $\cdots$ ③

- [Miyanishi84, §2.4] $\Rightarrow \mathfrak{R} := \mathfrak{A} \cap k[z, z^{-1}] = k[t]$ is a polynomial ring

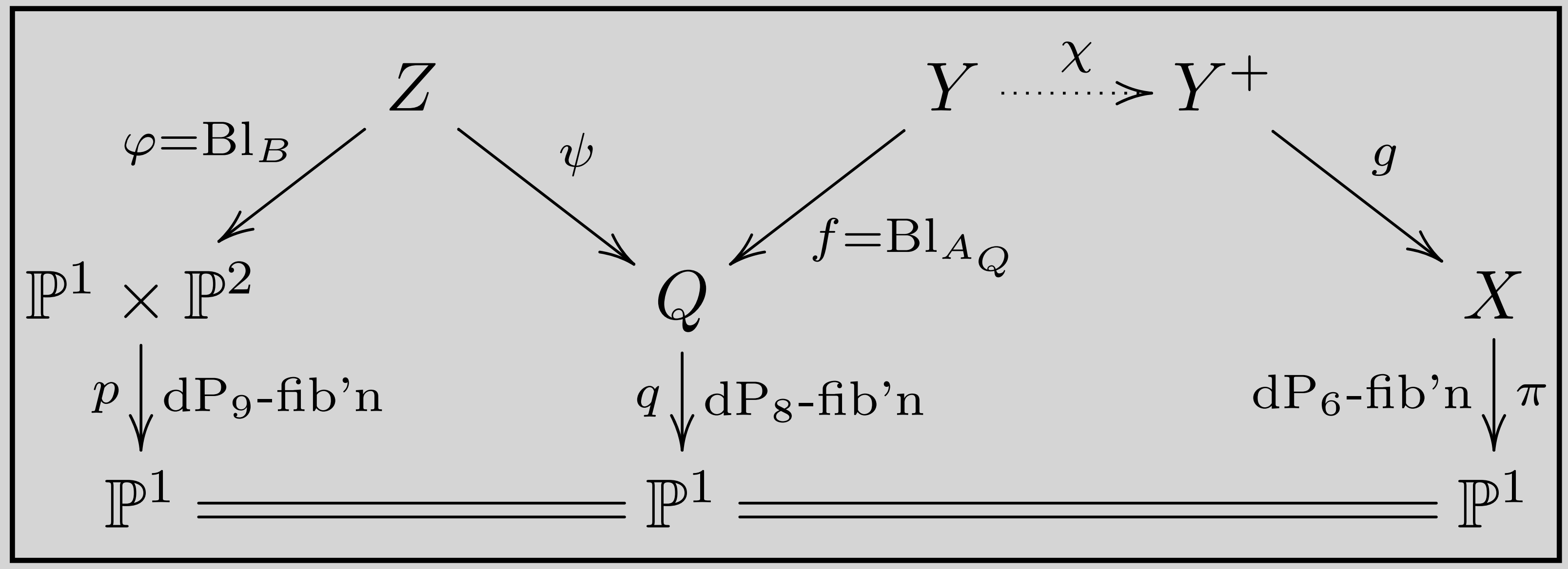
- ③ $\Rightarrow \forall$ fiber of p': $X \backslash (B_h \cup B_f) = \mathrm{Spec}(\mathfrak{A}) \to \mathrm{Spec}(\mathfrak{R}) \cong \mathbb{A}^1$ is affine plane

- [Sathaye83, Bass-Connell-Wright77] $\Rightarrow p'$ is trivial fibration

  $\Rightarrow X \backslash (B_h \cup B_f) \cong \mathbb{A}^3$ $\square$

Theorem E

(★)



- Let $\rho' : k^* \curvearrowright k^3; e \cdot (\alpha_3, \alpha_4, \alpha_7) = (e^3\alpha_3, e^2\alpha_4, e^4\alpha_7)$

  $\rho : k^* \curvearrowright k^4; e \cdot (\alpha_3, \alpha_4, \alpha_6, \alpha_7) = (e^3\alpha_3, e^2\alpha_4, e\alpha_6, e^4\alpha_7)$

- Fix $\vec{\alpha} = (\alpha_3, \alpha_4, \alpha_6, \alpha_7)$ and $\vec{\beta} = (\beta_3, \beta_4, \beta_6, \beta_7) \in k^4$,

- Set $\vec{\alpha}' = (\alpha_3, \alpha_4, \alpha_7)$ and $\vec{\beta}' = (\beta_3, \beta_4, \beta_7) \in k^4$,

- Then (1) $\vec{\alpha}' \sim_{\rho'} \vec{\beta}' \iff \exists \Phi \colon X_{\vec{\alpha}} \xrightarrow{\sim} X_{\vec{\beta}}$.

  (2) $\vec{\alpha} \sim_{\rho} \vec{\beta} \iff \exists \Phi \colon X_{\vec{\alpha}} \xrightarrow{\sim} X_{\vec{\beta}}$ s.t. $\Phi(B_{h,\vec{\alpha}}) = B_{h,\vec{\beta}}$ and $\Phi(B_{f,\vec{\alpha}}) = B_{f,\vec{\beta}}$

Proof of Theorem E

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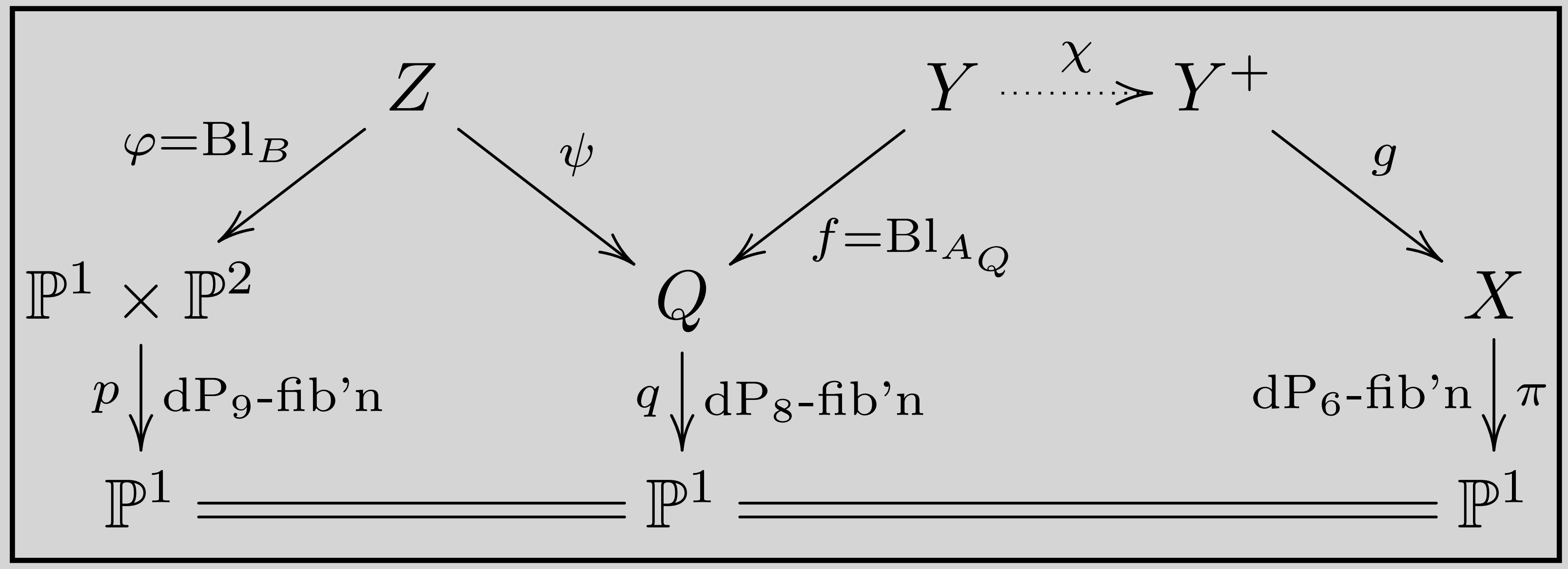

- **Claim :** $s_0 \subset X$ **is the unique** $(-K_X)$**-negative curve**

  ∵ For a curve $r \subset X$,

  $$r = s_0 \Rightarrow (-K_X \cdot r) = \frac{8(-K_Q \cdot T) - 24p_a(T) - (-K_Q)^3 - 32}{8} \quad \text{[Fukuoka17]}$$

  $$= -1$$

  $$r_{Y^+} = l^+ \Rightarrow (-K_X \cdot r) = (-K_{Y^+} + (H_1)_{Y^+} \cdot l^+) = ((H_1)_{Y^+} \cdot l^+) \geq 0$$

  Otherwise, $(-K_X \cdot r) = (-K_{Y^+} + (H_1)_{Y^+} \cdot r_{Y^+}) \geq (-K_{Y^+} \cdot r_{Y^+}) = (-K_Y \cdot r_Y) \geq 0$

Proof of Theorem E

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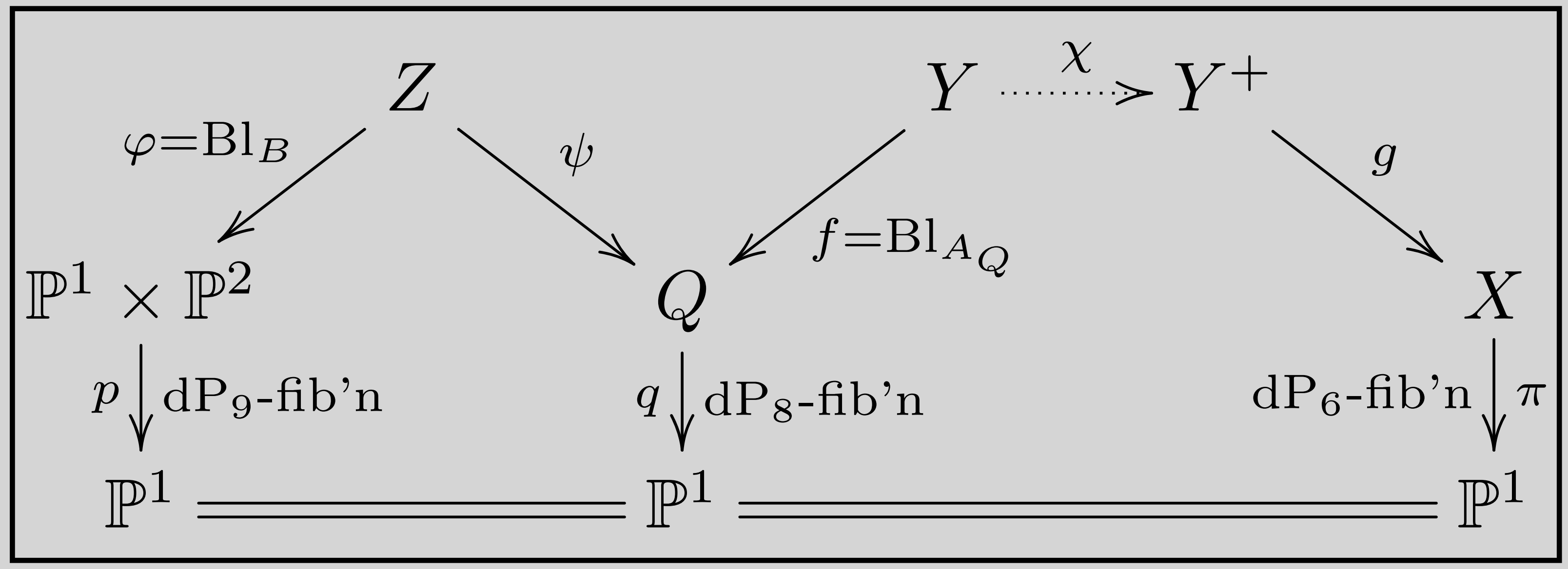


- Suppose $\exists \Phi \colon X_{\vec{\alpha}} \xrightarrow{\sim} X_{\vec{\beta}}$

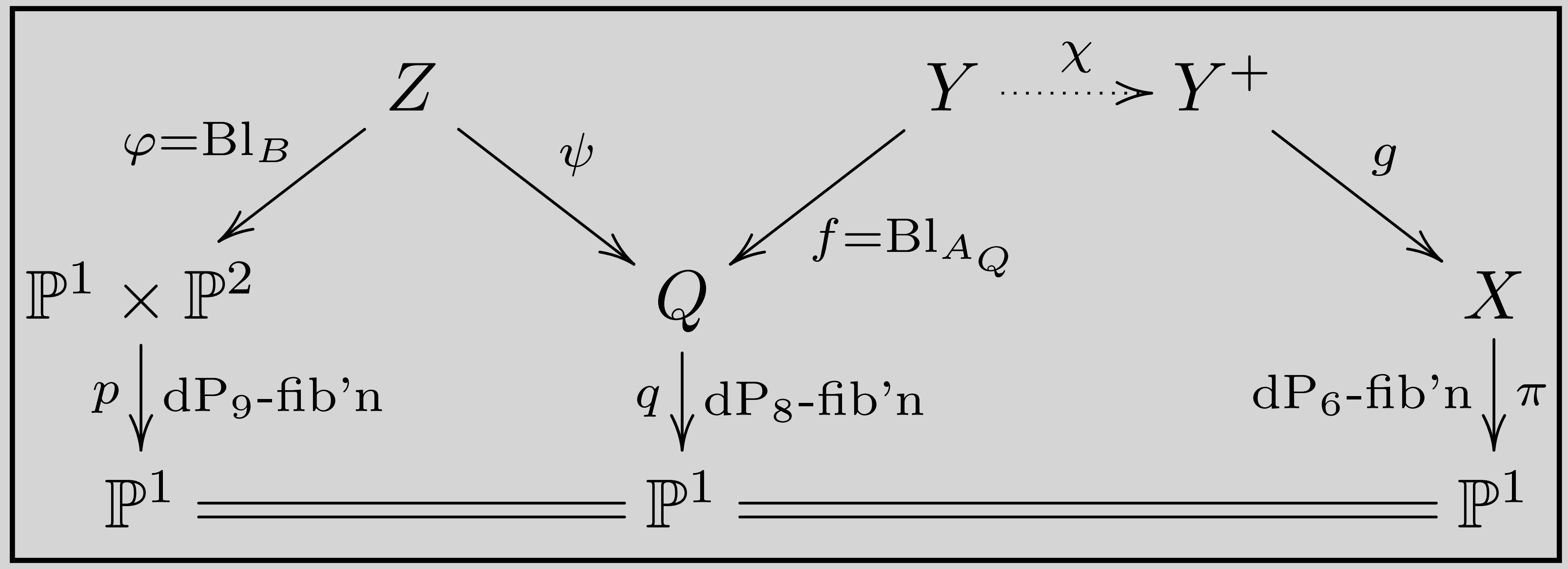
- $s_0 \subset X$ is the unique $(-K_X)$-negative curve

  $\Rightarrow \Phi(s_{0,\vec{\alpha}}) = s_{0,\vec{\beta}}$

  $\Rightarrow \Phi$ induces isomorphisms $Y^+_{\vec{\alpha}} \xrightarrow{\sim} Y^+_{\vec{\beta}}$, $Y_{\vec{\alpha}} \xrightarrow{\sim} Y_{\vec{\beta}}$, and $Q_{\vec{\alpha}} \xrightarrow{\sim} Q_{\vec{\beta}}$

  In particular, $Q_{\vec{\alpha}} \xrightarrow{\sim} Q_{\vec{\beta}}$ sends $T_{\vec{\alpha}}$ to $T_{\vec{\beta}}$

Proof of Theorem E

(★) 

- $s \subset Q$ is the unique $(-K_Q)$-trivial curve

  $\Rightarrow Q_{\vec{\alpha}} \xrightarrow{\sim} Q_{\vec{\beta}}$ sends $s_{\vec{\alpha}}$ to $s_{\vec{\beta}}$

  $\Rightarrow \Phi$ induces isomorphisms $Z_{\vec{\alpha}} \xrightarrow{\sim} Z_{\vec{\beta}}$, and $\Psi \colon \mathbb{P}^1 \times \mathbb{P}^2 \xrightarrow{\sim} \mathbb{P}^1 \times \mathbb{P}^2$

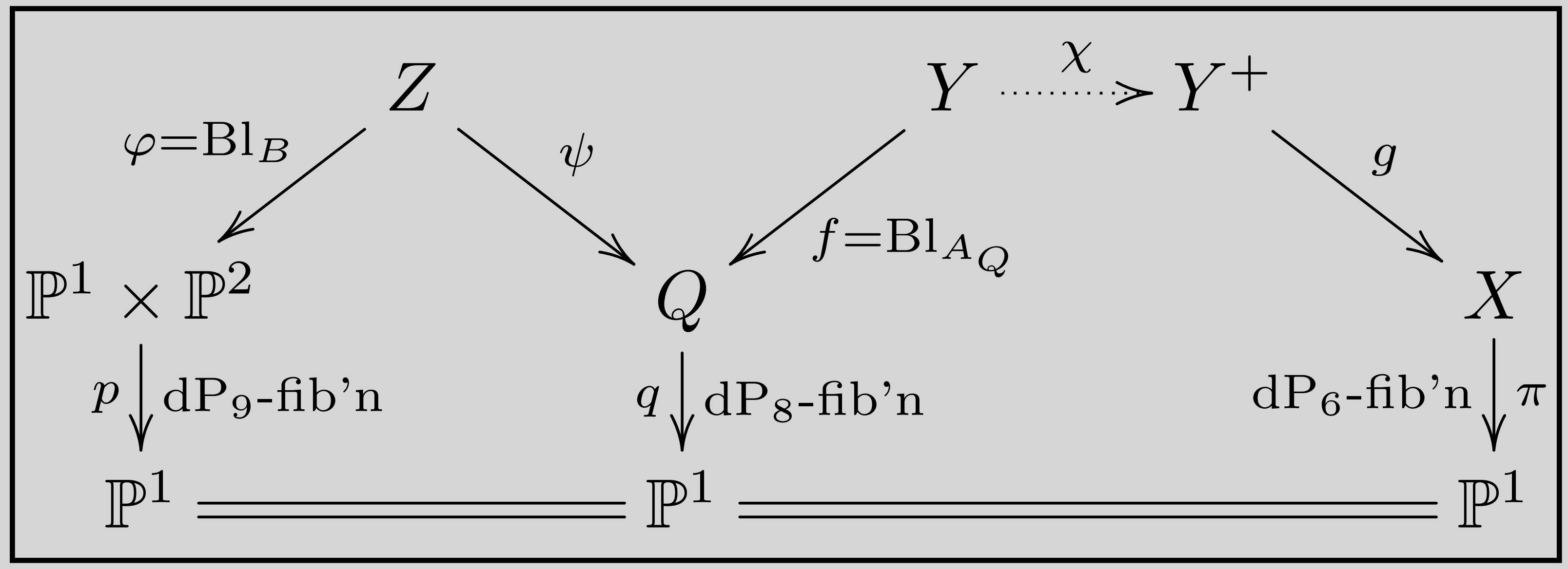
  In particular, Ψ sends $A_{\vec{\alpha}}$ (resp. $B_{\vec{\alpha}}$) to $A_{\vec{\beta}}$ (resp. $B_{\vec{\beta}}$)

- $\therefore\ \exists \Phi \colon X_{\vec{\alpha}} \xrightarrow{\sim} X_{\vec{\beta}} \iff \exists \Psi \in \mathrm{Aut}(\mathbb{P}^1 \times \mathbb{P}^2)$ s.t $\Psi(A_{\vec{\alpha}}) = A_{\vec{\beta}}$ & $\Psi(B_{\vec{\alpha}}) = B_{\vec{\beta}}$

  $\iff \overrightarrow{\alpha}' \sim_{\rho'} \vec{\beta}'$

- Proof of Theorem E (2) is similar.

Mukai 3-fold of $g = 12$

(★) 

- Y contains two $(-K_Y)$-trivial curves $l \sqcup s_Y$

- $N_l Y \cong \mathcal{O}_l(-1)^{\oplus 2}$ and $N_{s_Y} Y \cong \mathcal{O}_{s_Y}(-1)^{\oplus 2}$

- W (:= the anti-canonical model of Y) is a Fano 3-fold
  with two ODP
  of div. class rank three
  of Picard rank one (i.e. maximally non-factorial)
  with $(-K_W)^3 = (-K_Y)^3 = 22$ (i.e. Mukai 3-fold of genus 12)

Mukai 3-fold of $g = 12$

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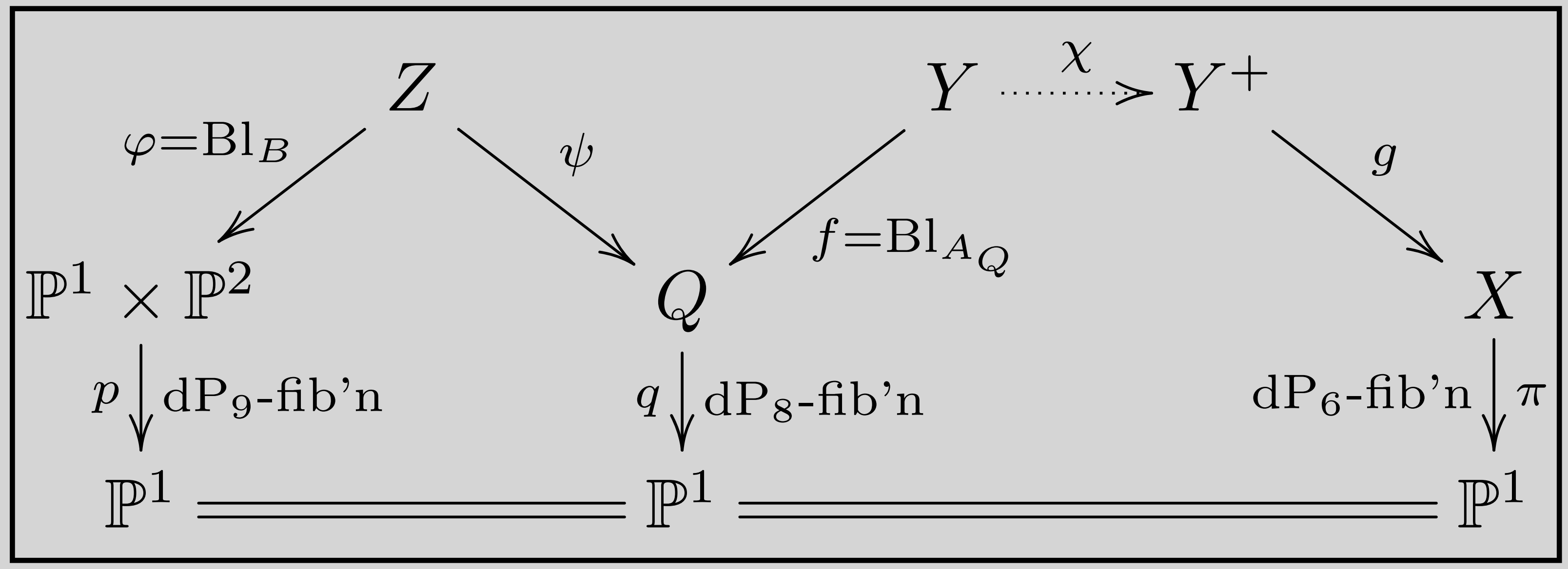
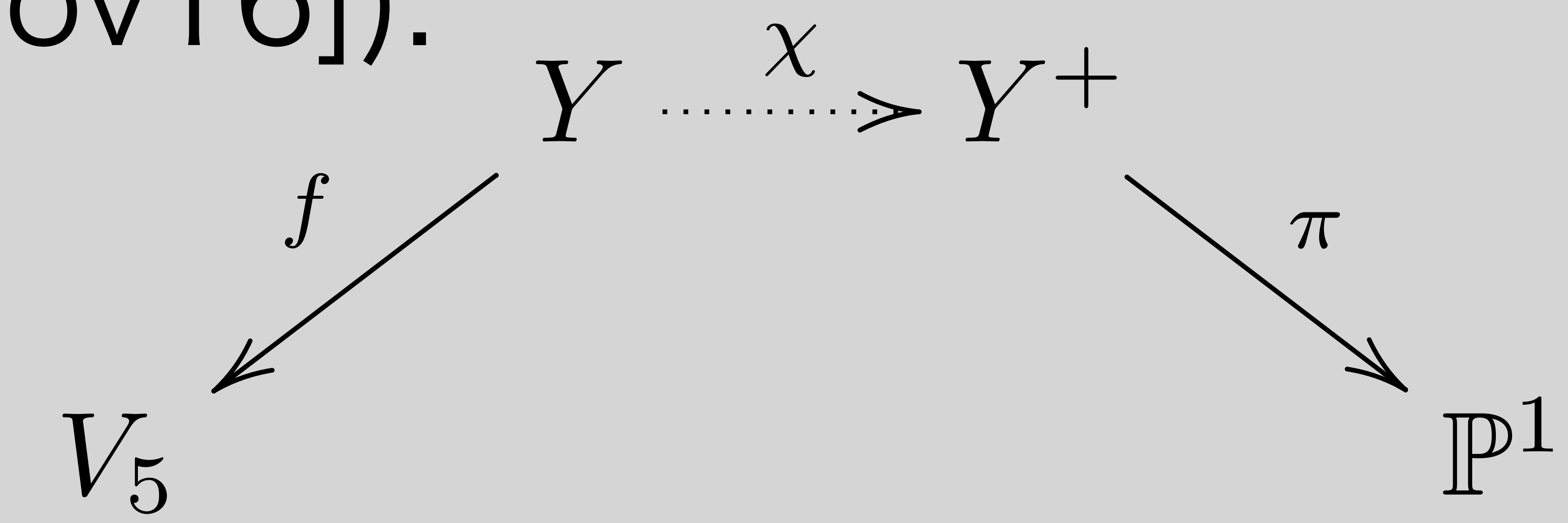


- [Mukai, Peternell]

  $\Rightarrow$ ∃4-dim'l family of smooth Mukai 3-folds of $g = 12 \supset \mathbb{A}^3$

- E.g. ([Prokhorov16]).



χ : Atiyah flop of Υ

π : del Pezzo fib'n of $d = 6$

Midpoint is Mukai 3-fold of $g = 12$ with one ODP $\supset \mathbb{A}^3$

- W = (Mukai 3-fold of $g = 12$ with two ODP) contains

$$Y \backslash ((S_h)_Y \cup (S_f)_Y \cup (H_1)_Y) \cong Y^+ \backslash ((S_h)_{Y^+} \cup (S_f)_{Y^+} \cup (H_1)_{Y^+} \cup l^+) \cong \mathbb{A}^1 \times (\mathbb{A}^2)^*$$

Thank you for listening