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Descending chain condition for finite morphisms of algebraic varieties

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§1 Preliminary results

k is an algebraically closed field of characteristic zero.
$k = \mathbb{C}$ when topological arguments are used.

Definition 1.1

The descending chain condition ((DCC), for short) for finite surjective morphisms of algebraic varieties belonging to a category $\mathcal{C}$ asserts that for any descending chain with X_i and f_i being objects and morphisms in $\mathcal{C}$,

$$X_1 \xrightarrow{f_1} X_2 \longrightarrow \cdots\cdots \to X_n \xrightarrow{f_n} X_{n+1} \longrightarrow \cdots$$

there exists an integer $N > 0$ such that f_n is an isomorphism for every $n \geq N$. We assume that a finite morphism is surjective.

Remark 1.2

(1) If an algebraic variety X has a finite endomorphism $f : X \to X$ of $\deg f > 1$ then the repetitions of f give a non-ending descending chain with all members isomorphic to X. Hence the (DCC) does not hold. This remark applies to abelian varieties and algebraic tori.
(2) Examples of algebraic variety not satisfying the (DCC) are the projective spaces $\mathbb{P}^n$ and the affine spaces $\mathbb{A}^n$. In fact, if $\{X_0, X_1, \ldots, X_n\}$ is a system of homogeneous coordinates of $\mathbb{P}^n$ the morphism

$$\{X_0, X_1, \ldots, X_n\} \mapsto \{X_0^m, X_1^m, \ldots, X_n^m\}, \quad m > 1$$

gives a finite endomorphism of $\mathbb{P}^n$ with degree m^n. Similarly, if $\{x_1, \ldots, x_n\}$ is a system of coordinates of $\mathbb{A}^n$, the morphism $\{x_1, \ldots, x_n\} \mapsto \{x_1^m, \ldots, x_n^m\}$ gives a finite endomorphism of $\mathbb{A}^n$ of degree m^n.

We study if the (DCC) for finite morphisms holds in the cases:

(1) All X_i are smooth varieties of non-negative Kodaira dimension.

(2) All X_i are smooth $\mathbb{Q}$-homology planes, which we call $\mathbb{Q}$-*planes*.

(3) There exists an algebraic group G such that all X_i are G-varieties and all f_i are G-equivariant.

(4) All X_i are del Pezzo surfaces.

First consider the case (1) for curves. Let

$$X_1 \xrightarrow{f_1} X_2 \longrightarrow \cdots\cdots \longrightarrow X_n \xrightarrow{f_n} X_{n+1} \longrightarrow \cdots \qquad (1)$$

be an infinite descending chain of finite morphisms of normal algebraic curves with $d_i := \deg f_i > 1$ for every i.

Let C_i : normal completion of X_i and
$\varphi_i : C_i \to C_{i+1}$ extension of f_i.
We have an infinite descending chain of finite morphisms of complete normal curves

$$C_1 \xrightarrow{\varphi_1} C_2 \longrightarrow \cdots\cdots \to C_n \xrightarrow{\varphi_n} C_{n+1} \longrightarrow \cdots . \tag{2}$$

$D_i = C_i - X_i$, a reduced divisor, $m_i = \deg D_i$. If X_1 is complete then $X_i = C_i$ and $m_i = 0$ for all i. Since $f_i : X_i \to X_{i+1}$ is a finite morphism, $\varphi_i^*(D_{i+1})_{\mathrm{red}} = D_i$. By the log ramification divisor formula, we have

$$K_{C_i} + D_i \sim \varphi_i^*(K_{C_{i+1}} + D_{i+1}) + R_i + S_i, \tag{3}$$

where $R_i + S_i$ is the log ramification divisor with $\mathrm{Supp}\,(S_i) \subset D_i$ and $\mathrm{Supp}\,(R_i) \subset X_i$.

Claim $S_i = 0$ if X_1 is affine. In fact, let Q be a point of D_{i+1} with a local parameter y of C_{i+1} at Q. Write ${\varphi_i}^*(y) = c \prod_j x_j^{e_j}$, where x_j is a local parameter of C_i at a point of D_i over Q. Then ${\varphi_i}^*(dy/y) = \sum_j e_j(dx_j/x_j)$. So, $S_i = 0$. Let g_i be the genus of C_i. Then we have

$$2g_i - 2 + m_i = d_i(2g_{i+1} - 2 + m_{i+1}) + \deg R_i, \qquad (4)$$

where $\deg R_i \geq 0$. Since $d_i \geq 2$ and $d_i m_{i+1} \geq m_i$ as the inverse image of a point of D_{i+1} by φ_i has at most d_i points, we have

$$2g_i - 2 = d_i(2g_{i+1} - 2) + (d_i m_{i+1} - m_i) + \deg R_i \geq 2(2g_{i+1} - 2). \quad (5)$$

Hence $g_i \geq 2g_{i+1} - 1$, whence $g_i \geq g_{i+1}$. Also $m_i \geq m_{i+1}$. The sequences $\{g_i\}$ and $\{m_i\}$ are descending sequences of non-negative integers. Replacing the given chain by a subchain, which is obtained from the original chain by removing finitely many terms, we may assume $g_i = g_{i+1}$ and $m_i = m_{i+1}$ for every i. So we have

$$0 = (d_i - 1)(2g_i - 2 + m_i) + \deg R_i. \qquad (6)$$

Since $m_i \geq 0$, $d_i \geq 2$ and $\deg R_i \geq 0$, one of the following cases takes place.

(i) $g_i = 1$, $m_i = 0$ and $\deg R_i = 0$. Then X_i is an elliptic curve and f_i is unramified.

(ii) $g_i = 0$, $m_i = 0$ and $\deg R_i = 2(d_i - 1)$. Then $X_i \cong \mathbb{P}^1$ and f_i is a cyclic covering of degree d_i.

(iii) $g_i = 0$, $m_i = 1$ and $\deg R_i = d_i - 1$. Then $X_i \cong \mathbb{A}^1$ and f_i is a cyclic covering of degree d_i.

(iv) $g_i = 0$, $m_i = 2$ and $\deg R_i = 0$. Then $X_i \cong \mathbb{A}^1_*$ and f_i is a cyclic covering of degree d_i.

Lemma 1.3

For a descending chain (1) for normal algebraic curves with $d_i := \deg f_i > 1$, we have:

(1) If X_1 is complete then either $X_n \cong \mathbb{P}^1$ or X_n is an elliptic curve for $\forall n \geq N$ for $\exists N$.

(3) If X_1 is affine then either $X_n \cong \mathbb{A}^1$ or $X_n \cong \mathbb{A}^1_*$ for $\forall n \geq N$ for $\exists N$.

§2 Homology planes : $\mathbb{Z}$-planes and $\mathbb{Q}$-planes

Lemma 2.1

Let $f : X \to Y$ be a finite morphism of affine normal surfaces of degree d. If X is a $\mathbb{Z}$-plane (resp. $\mathbb{Q}$-plane), so is Y.

Proof.

Suppose $H_i(X;\mathbb{Q}) = 0$ for $i = 1, 2$. Since $f_* : H_i(X;\mathbb{Q}) \twoheadrightarrow H_i(Y;\mathbb{Q})$ is surjective for $i = 1, 2$, Y is a $\mathbb{Q}$-plane. $H_2(Y;\mathbb{Z})$ is torsion-free if $H_2(Y;\mathbb{Z}) \neq 0$ by Theorem of Kaup-Narasimhan-Hamm. If X is a $\mathbb{Z}$-plane, $d\xi = 0$ for $\xi \in H_2(Y;\mathbb{Z})$. So, $H_2(Y;\mathbb{Z}) = 0$.

Let $\xi \in H_1(Y;\mathbb{Z})$ be a nonzero element. Then $d\xi = 0$. If $\xi \neq 0$, ξ corresponds to an abelian Galois extension $\sigma : Y' \to Y$, where σ is étale. Since $H_1(X;\mathbb{Z}) = 0$, f splits as $f : X \xrightarrow{f'} Y' \xrightarrow{\sigma} Y$, where f' is a finite morphism and Y' is a $\mathbb{Q}$-plane. Then $\chi(Y') = 1$. But $\chi(Y') = (\deg\sigma)\chi(Y)$ as σ is étale and finite. So, $\deg\sigma = 1$. This is a contradiction. Hence Y is a $\mathbb{Z}$-plane. □

X : a $\mathbb{Q}$-plane,
$X \hookrightarrow V$ a log smooth completion,
$D = V - X$ an SNC-divisor, D a tree of $\mathbb{P}^1$s.
N : a tubular nbd of D s.t. D is a strong deformation retract of N,
$M = \partial N$: the boundary of N. Then M is a compact orientable 3-mfd, called a 3-mfd at infinity of X.
$\pi_1^\infty(X) := \pi_1(M)$: the *fundamental group at infinity of* X, which is described by generators and relations with repsect to the intersection matrix of D (the *Mumford-Ramanujam method*).

Lemma 2.2

Let X be a complex affine normal surface.

(1) X is isomorphic to $\mathbb{A}^2$ if and only if $\pi_1^\infty(X) = (1)$.

(2) Suppose X is topologically contractible. Then X is isomorphic to $\mathbb{A}^2/G$ if and only if $\pi_1^\infty(X)$ is a finite group, where $G \cong \pi_1^\infty(X)$.

Lemma 2.3

Let X be a $\mathbb{Z}$-plane (resp. $\mathbb{Q}$-plane) and let M be a 3-manifold at infinity. Then M is a $\mathbb{Z}$-homology (resp. $\mathbb{Q}$-homology) 3-sphere.

Proof.

Let $X \hookrightarrow V$ be a minimal log smooth completion of X and $D = V - X$. Since X is factorial, $\mathrm{Pic}\,(V)$ is generated freely by the components of D. Since X is rational, $p_g(V) = 0$. So all the 2-cycles on V are algebraic. Note that if X is a $\mathbb{Q}$-plane then the determinant of the intersection matrix of D is non-zero. This implies, by Poincaré duality, that the intersection form on the components of D is unimodular if X is a $\mathbb{Z}$-plane. Finally, D is a tree of the $\mathbb{P}^1$. $H_1(M;\mathbb{Z}) = 0$ since the intersection form on D is unimodular. Hence M is a $\mathbb{Z}$-homology 3-sphere. All this argument works with $\mathbb{Q}$-homologies if X is a $\mathbb{Q}$-plane and in this case the 3-manifold M is a $\mathbb{Q}$-homology 3-sphere. □

Lemma 2.4

Let $f : X \to Y$ be a finite morphism of $\mathbb{Z}$-planes. Then f induces a surjective homomorphism $f_* : \pi_1^\infty(X) \twoheadrightarrow \pi_1^\infty(Y)$.

Proof.

It can be shown that f induces a group homomorphism $f_* : \pi_1^\infty(X) \to \pi_1^\infty(Y)$ such that the image H of f_* is a subgroup of finite index. Further, f has a factorization $f : X \xrightarrow{f'} Y' \xrightarrow{f''} Y$ such that $\pi_1^\infty(Y') \cong H$, f' is a finite morphism and f'' is a finite étale morphism. By Lemma 2.1, Y' is a $\mathbb{Z}$-plane. Hence $\chi(Y') = 1$ and $\chi(Y') = \deg f'' \cdot \chi(Y) = \deg f''$. So, f'' is an isomorphism. This implies that f_* is surjective. □

The following result is known in differential topology.

Lemma 2.5

Any closed orientable 3-manifold dominates only finitely many $\mathbb{Z}$-homology 3-spheres.

Related to these results, the following problem is interesting.

Problem 2.6

Let $f : X \to Y$ be a finite morphism of smooth affine surfaces such that the induced homomorphism $f_* : \pi_1^\infty(X) \to \pi_1^\infty(Y)$ is an isomorphism, where we assume that $\pi_1^\infty(X) \neq (1)$. Is f then an isomorphism?

There is an affirmative partial answer to Problem 2.6.

Theorem 2.7

Let $f : X \to Y$ be a finite morphism of smooth affine surfaces X and Y with $\overline{\kappa}(X) \geq 0$ and $\overline{\kappa}(Y) \geq 0$. Suppose that the induced homomorphism $f_* : \pi_1^\infty(X) \to \pi_1^\infty(Y)$ is an isomorphism. Then f is an isomorphism.

Proof.

Let M, N be the 3-mfds at infinity for X, Y respectively. By Perelman's result both M, N are geometric 3-mfds. Clearly, M, N are orientable. Let $X \subset V$, $Y \subset W$ be minimal log smooth completions and let $D := V - X$, $\Delta := W - Y$. Then any maximal rational twig of D (resp. Δ) is admissible. By a result of W. Neumann, M, N are both prime manifolds. Since $\pi_1^\infty(M) \xrightarrow{\sim} \pi_1^\infty(N)$, by W. Neumann, M and N are homeomorphic. It is then shown that f is either a homotopy equivalence, or homotopic to a topological covering map unless $\pi_1(M)$ is finite or cyclic.

Suppose $\deg f > 1$. Then the following assertions hold.

(1) $\pi_1(M)$ is not finite.

(2) $\pi_1(M)$ is not cyclic.

(3) $f : M \to N$ is not a homotopy equivalence.

(4) $f : M \to N$ is not homotopic to a topological covering.

So, we have $\deg f = 1$, and f is an isomorphism. □

There is an unsolved problem asking if a smooth affine hypersurface $S := \{x^2 + y^2 + z^2 = 1\}$ has a finite endomorphism of degree > 1.

Proposition 2.8

Suppose that Problem 2.6 holds in the case $\overline{\kappa} = -\infty$. Then we have:

(1) $\pi_1^\infty(S) \cong \mathbb{Z}/2\mathbb{Z}$.

(2) There are no finite endomorphisms of S with degree > 1.

Proof.

(2) Let $f : S \to S$ be a finite endomorphism of degree > 1. Then f induces a group endomorphism f_* of $\pi_1^\infty(S) \cong \mathbb{Z}/2\mathbb{Z}$. If f_* is an isomorphism then f is an isomorphism by the assumption. Otherwise, f_* has image (0). Then there is a splitting $f : S \xrightarrow{f'} X \xrightarrow{f''} S$, where X is a smooth affine surface with $\pi_1^\infty(X) = (0)$. By Lemma 2.2, $X \cong \mathbb{A}^2$. But there is no finite morphism $\mathbb{A}^2 \to S$. □

§3 G-varieties

Problem 3.1

Let G be an algebraic group defined over k. Let

$$X_1 \xrightarrow{f_1} X_2 \longrightarrow \cdots\cdots \longrightarrow X_n \xrightarrow{f_n} X_{n+1} \longrightarrow \cdots \tag{7}$$

be a descending chain of normal affine G-varieties and G-equivariant finite morphisms. Assume that the algebraic quotient $Y_i = X_i/G$ exists as an algebraic variety for all i. Suppose that

$$Y_1 \xrightarrow{g_1} Y_2 \longrightarrow \cdots\cdots \longrightarrow Y_n \xrightarrow{g_n} Y_{n+1} \longrightarrow \cdots \tag{8}$$

satisfies the (DCC). Does the chain (7) satisfy the (DCC)?

Here we note the following result.

Lemma 3.2

Let $A \supset B$ be normal affine k-domains such that A is a finite B-module. We assume that G acts on A and B equivariantly. Let A_0 and B_0 be the rings of G-invariants of A and B, respectively. Then A_0 is integral over B_0. If A_0 is an affine k-domain then A_0 is a finite B_0-module and B_0 is an affine k-domain.

Proof.

Let z be an element of A_0. The minimal equation of z over $Q(B)$, $f(z) = z^n + \beta_1 z^{n-1} + \cdots + \beta_n = 0,\ \beta_i \in Q(B)$, is a monic equation over B if B is normal. Hence A_0 is integral over $B_0 = A_0 \cap B$. □

Proposition 3.3

(1) If G is a finite group, Problem 3.1 has a positive answer.

(2) If G is a reductive algebraic group of positive dimension, Problem 3.1 has a negative answer.

Proof.

Replace the chain (7) by a subchain and assume that g_i are all isomorphisms. Suppose G is reductive. Then each quotient Y_i is a *good* quotient. So, there exists an open set U_i of Y_i such that $q_i^{-1}(Q)$ is a closed orbit isomorphic to G/H_i for $Q \in U_i$. Let P_1 be a point of X_1 s.t. $GP_1 \cong q_1^{-1}(q_1(P_1)) \cong G/H_1$. Let $P_i = (f_{i-1} \circ \cdots \circ f_1)(P_1)$ for $i \geq 2$ and $q_i^{-1}(q_i(P_i)) \cong G/H_i$ with the stabilizer group H_i of P_i. So we have an ascending chain of subgroups of G,

$$H_1 \subset H_2 \subset \cdots \subset H_i \subset \cdots \quad . \tag{9}$$

(1) If G is a finite group. Then $H_n = H_N$ for every $n \geq \exists N$. Then $f_n : X_n \to X_{n+1}$ is birational for $n \geq N$, and hence an isomorphism by the Zariski Main Theorem.
(2) If G is reductive of $\dim G > 0$. Then G contains G_m as a subgroup. Hence there exists an infinite ascending chain of cyclic finite subgroups of G; $H_1 \subsetneq H_2 \subsetneq \cdots\cdots \subsetneq H_n \subsetneq H_{n+1} \subsetneq \cdots$. Let $X_i = G/H_i$ be the homogeneous space of the left cosets of H_i. Then the chain $\{X_i\}$ never stops. □

Theorem 3.4

In Problem 3.1, assume that $G = G_a$ and the X_i are affine normal G_a-varieties. If the chain (8) satisfies (DCC) then the chain (7) satisfies the (DCC).

Proof.

It is known that the image of the quotient morphism $q_i : X_i \to Y_i$ contains all codimension one points of Y_i for every i . Suppose that in the chain (8), $g_n : Y_n \to Y_{n+1}$ is an isomorphism for every $n \geq N$. Since the G_a-action on X_n is non trivial there exists an open set U_n of Y_n such that $q_n^{-1}(U_n) \cong U_n \times \mathbb{A}^1$ and G_a acts along the fiber $\mathbb{A}^1$. Since this is the case for $q_{n+1} : X_{n+1} \to Y_{n+1}$, we may assume that g_n induces an open immersion $U_n \hookrightarrow U_{n+1}$. Then the fiberwise G_a-action induces an isomorphism of the fibers of q_n and q_{n+1}. Hence f_n is birational. This implies that f_n is an isomorphism by the Zariski Main Theorem. □

§4 Affine algebraic surfaces with $\overline{\kappa} \geq 0$

Let $f : X \to Y$ be a finite morphism of smooth affine surfaces. Then there exist log smoooth completions V, W of X, Y such that the morphism f extends to a morphism $\Phi : V \to W$ with $\Phi|_X = f$ and $\Phi^{-1}(\Delta) = D$, where $D = V - X$ and $\Delta = W - Y$. By the log ramification formula, we have $D + K_V = \Phi^*(\Delta + K_W) + R$, where R is the log ramification divisor ≥ 0.

Lemma 4.1

R is supported by the union of curves C on V such that

(1) C is contracted by Φ (hence a component of D). The component C of D has coefficient zero in R if C is a component not contracted by Φ.

(2) $C \cap X \neq \emptyset$ and $\Phi|_C : C \to \Phi(C)$ is ramified.

(3) If a component C of D has coefficient zero in R and C is contracted by Φ, then it is contracted to an intersection point of two irreducible components of Δ.

Write the log ramification divisor R as $R = R_1 + R_2$, where R_1 is the sum of irreducible components meeting X and R_2 is the sum of componets of D. Since R_2 is contracted by Φ, mR_2 is in the base locus of the linear system $|m(D + K_V)|$ for $\forall m > 0$.

Lemma 4.2

Suppose that $\dim |m(D + K_V)| = \dim |m(\Delta + K_W)| > 0$ for some positive integer m, then mR_1 is contained in the base locus of $|m(D + K_V)|$. The converse also holds. Hence this condition depends only on the minimal log smooth completions of X and Y.

Proof.

It is clear because
$|m(D + K_V)| = |\Phi^*(m(\Delta + K_W)) + mR_1 + mR_2| =$
$|\Phi^*(m(\Delta + K_W)) + mR_1| = \Phi^*(|m(\Delta + K_W)|)$, where
$\Phi^*(|m(\Delta + K_W)|)$ is the set of pull-backs by Φ of all members of $|m(\Delta + K_W)|$. The assumption implies that the movable part of $|m(D + K_V)|$ comes from the one for $\Phi^*(|m(\Delta + K_W)|)$. □

Let (V_0, D_0) be a minimal log smooth completion of X obtained by the contraction $\sigma_V : V \to V_0$ and let (W_0, Δ_0) be a minimal log smooth completion of Y obtained by $\sigma_W : W \to W_0$. Then $\Phi_0 := \sigma_W \circ \Phi \circ {\sigma_V}^{-1} : V_0 \dashrightarrow W_0$ is a dominant rational map which restricts to $f : X \to Y$. Then $\dim |m(D_0 + K_{V_0})| = \dim |m(D + K_V)|$ and $\dim |m(\Delta_0 + K_{W_0})| = \dim |m(\Delta + K_W)|$. Hence $\dim |m(D_0 + K_{V_0})| = \dim |m(\Delta_0 + K_{W_0})|$ if and only if $\dim |m(D + K_V)| = \dim |m(\Delta + K_W)|$.

Theorem 4.3

Let the chain (1) consist of smooth affine surfaces of the same log Kodaira dimension $\overline{\kappa}(X_i) \geq 0$ and (V_i, D_i) be a minimal log smooth completion of X_i for every i. Then the following assertions hold.

(1) If $\overline{\kappa}(X_i) = 2$ then (DCC) holds for (1).

(2) If $\overline{\kappa}(X_i) = 0$ or 1 then either (DCC) holds or $\chi(X_i) = 0$ for $i \gg 0$.

Proof.

By Tsunoda, $\dim|m(D_i + K_{V_i})| \geq 0$ for $m \geq 12$ and every i. Further, $\dim|m(D_i + K_{V_i})| \geq \dim|m(D_{i+1} + K_{V_{i+1}})|$ for every i. So, we may assume that $\dim|m(D_i + K_{V_i})| = \dim|m(D_{i+1} + K_{V_{i+1}})|$ for all $i \geq 1$. Let R_{1i} be the log ramification divisor of $V_1 \dashrightarrow V_i$ whose components have nonempty intersection with X_1. By Lemma 4.2, the divisor mR_{1i} is contained in the base locus of $|m(D_1 + K_{V_1})|$. Furthermore, if $j \geq i$ then $mR_{1j} \geq mR_{1i}$. Since mR_{1i} is bounded by the base locus of $|m(D_1 + K_{V_1})|$, $mR_{1n} = mR_{1N}$ for $\forall n \geq \exists N$. Hence $f_n : X_n \to X_{n+1}$ is étale for $n \geq N$. Then $\chi(X_n) = (\deg f_n)\chi(X_{n+1}) = \deg f_n)(\deg f_{n+1})\chi(X_{n+2}) = \cdots$. If $\overline{\kappa}(X_i) = 2$ then $\chi(X_n) > 0$. Hence $\deg f_n = \deg(f_{n+1}) = \cdots = 1$. Then f_n is birational and hence an isomorphism. If $\overline{\kappa}(X_n) = 0$ or 1 and the (DCC) fails, we must have $\chi(X_n) = 0$. □

Theorem 4.3 can be generalized to the higher-dimensional case. The proof is essentially the same as in the surface case.

Theorem 4.4

Let the chain (1) be a descending chain of smooth quasi-projective varieties of the same log Kodaira dimension $\overline{\kappa}(X_i) \geq 0$ and let (V_i, D_i) be a log smooth completion of X_i for every i. Suppose that there exists an integer m_0 such that $|m_0(D_i + K_{V_i})| \neq \emptyset$ for all i. If $\chi(X_i) > 0$ for all i, then the (DCC) holds.

§5 Del Pezzo surfaces

Lemma 5.1

Let $\varphi : V \to W$ be a finite morphism of del Pezzo surfaces with degree $d \geq 1$. Then we have.

(1) $\rho(V) \geq \rho(W)$.

(2) Suppose $\rho(V) = \rho(W)$ and $V \not\cong \mathbb{F}_0$. We keep this assumption except for (6). Then, for any (-1)-curve E on W, $\varphi^*(E) = mC$ with a (-1)-curve C on V, where $d = m^2$ and $\varphi|_C : C \to E$ is a finite morphism of degree m.

(3) Let R be the ramification divisor for φ. Then $R \geq \sum_E (m-1)E$.

(4) $d = 1$ and φ is an isomorphism if $\rho(V) \geq 5$.

(5) If $\rho(V) \leq 4$ then φ is induced by a finite endomorphism of $\mathbb{P}^2$ of degree $d = m^2$.

(6) Suppose $V \cong \mathbb{F}_0$. Then V has an endomorphism of arbitrary degree $d > 1$.

Proof of (1), (2), (3)

(1) By the projection formula, the natural homomorphism $\mathrm{Pic}\,(W) \otimes \mathbb{Q} \to \mathrm{Pic}\,(V) \otimes \mathbb{Q}$ is injective. Hence $\rho(W) \le \rho(V)$.

(2) For a (-1)-curve C on W, $\varphi^{-1}(C)$ is a divisor with negative-definite intersection form (Hodge index theorem). So, each component E of $\varphi^{-1}(C)$ is a (-1)-curve because $(E^2) < 0$. If $\varphi^{-1}(C) \ge E + E'$ with $E \ne E'$ then $E \cap E' = \emptyset$, for otherwise $(E + E')^2 \ge 0$. Further, E and E' are numerically independent in $\mathrm{Pic}\,(V)$. Hence $\rho(V) \ge \rho(W) + 1$, which contradicts the assumption. So, $\varphi^*(C) = mE$ and $(m-1)E$ is contained in the ramification locus R. Let s be the degree of $\varphi|_E : E \to C$ which is a cyclic covering. Then $(\varphi^*(C)^2) = (\varphi_*\varphi^*(C) \cdot C) = (\varphi_*(mE) \cdot C) = m(sC \cdot C) = -ms = -d$ and $(\varphi^*(C)^2) = m^2(E^2) = -m^2$. Hence $s = m$ and $d = m^2$.

(3) Since $\rho(V) = \rho(W)$, the number of (-1)-curves in V and W is the same. So, each (-1)-curve E in V is the reduced inverse image $\varphi^{-1}(C)$ with a (-1)-curve C in W. So, R contains a divisor $(m-1)\sum_E E$, which is the sum of all (-1)-curves in V.

Proof of (4)

(4) Let $V \cong V_n := \mathrm{Bl}_{P_1,\dots,P_n}\mathbb{P}^n$. Assume that $n \geq 2$ so that there exists a (-1)-curve which is the proper transform of a line ℓ on $\mathbb{P}^2$. Then we can choose mutually disjoint (-1)-curves E_i $(1 \leq i \leq n)$ such that the contraction, say σ, of the E_i maps V to $\mathbb{P}^2$. This is the case for W and mutually disjoint (-1)-curves C_i $(1 \leq i \leq n)$, where $C_i = \varphi(E_i)$. Let τ be the contraction of the C_i. Then φ induces a finite morphism $\psi : \mathbb{P}^2 \to \mathbb{P}^2$ of degree m^2 such that $\tau \circ \varphi = \psi \circ \sigma$. Let ℓ (resp. L) be a line on the source (resp. target) $\mathbb{P}^2$ whose proper transform $\sigma'(\ell)$ (resp. $\tau'(L)$) on V (resp. W) is a (-1)-curve. We may assume that $\psi(\ell) = L$. Let N be the number of (-1)-curves on V_n. Since $K_V \sim \varphi^*(K_W) + R$, we have $K_{\mathbb{P}^2} \sim \psi^*(K_{\mathbb{P}^2}) + S$, where $S = \sigma_*(R)$ is the ramification locus of ψ and S contains the sum of images of (-1)-curves E on V with coefficient $m-1$. Since $\psi^*(K_{\mathbb{P}^2}) = \psi^*(-3L)$ and the ramification index of ℓ over L is equal to m by the above choice of ℓ and L, we have $K_{\mathbb{P}^2} \sim -3\ell$ and $\psi^*(K_{\mathbb{P}^2}) \sim \psi^*(-3L) = -3m\ell$, whence $-3\ell \sim -3m\ell + S$ with $S \geq (m-1)N_1\ell$, where N_1 is the number of (-1)-curves in V which come from lines on $\mathbb{P}^2$.

Proof of (4) continued, (5), (6)

Hence we have

$$3(m-1) \geq (m-1)N_1. \tag{10}$$

We have the following list

n	0	1	2	3	4	5	6	7	8
(K_V^2)	9	8	7	6	5	4	3	2	1
N_1	0	0	1	3	6	10	15	21	28
N	0	1	3	6	10	16	27	56	240

By the table and the equation (10), $m = 1$ if $n \geq 4$. Namely φ is an isomorphism.

(5) By the argument in (4), if $\rho(V) \leq 4$, then V is obtained from $\mathbb{P}^2$ by blowing up at most three points. The assertion follows immediately.

(6) Since $\mathbb{P}^1$ has a finite endomorphism α_m of degree m, $(X_0, X_1) \mapsto (X_0^m, X_1^m)$. So, $\mathbb{F}_0$ has an endomorphism $\alpha_m \times \alpha_n$.

The following theorem follows from Lemma 5.1.

Theorem 5.2

Let

$$V_1 \xrightarrow{\varphi_1} V_2 \longrightarrow \cdots\cdots \longrightarrow V_n \xrightarrow{\varphi_n} V_{n+1} \longrightarrow \cdots$$

be a descending chain of del Pezzo surfaces with the same Picard number $\rho(V_i) = \rho$. If $5 \leq \rho \leq 9$, the chain satisfies the (DCC). If $1 \leq \rho \leq 4$ the chain does not necessarily satisfy the (DCC).